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Classification and Properties of Dual Conical Congruences.

Inaugural-Dissertation
zur
Erlangung der philosophischen Doktorwürde
vorgelegt der
Hohen philosophischen Fakultät
(Mathematisch-naturwissenschaftliche Sektion)
der
UNIVERSITÄT ZÜRICH
von

Archie Burton Pierce
aus **Berkeley**, Californien, U. S. A.

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Begutachtet von
Herrn Prof. Dr. H. Burkhardt.

Zürich 1903.
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Introduction.

In the „Geometrie der Dynamen“,* Study has introduced the system of complex quantities which he calls dual quantities and which are represented by

$$\mathfrak{x} = x + \xi\varepsilon,$$

where $\varepsilon^2 = 0$, and x and ξ are ordinary complex quantities and are called scalars. He then shows that the ratio of three such quantities

$$X_1 : X_2 : X_3$$

can be used as the coordinates (homogeneous in this dual number system) of the elements of a four fold manifoldness which contains all the elements of the four-fold manifoldness resulting from considering the ordinary three-dimensional space with the straight line as element as given in the Plücker Line-geometry. In fact the two domains coincide except for a certain restricted portion of the new manifoldness which has no corresponding elements in the Plücker manifoldness. In this way a system of coordinates is obtained for the elements of this higher manifoldness, which is very closely analogous to that used in the analytic geometry of the plane. They are called Ray-coordinates and the corresponding geometry is called Ray-geometry.

In the geometry of the plane, we have the two following methods of determining the various curved loci analytically; 1) that in which the coordinates of its elements satisfy an equation; and 2) that in which the coordinates of the elements are given as functions of a single parameter. From the above-mentioned analogy, we are then led to ask concerning the nature of the configurations represented in the corresponding way in the new geometry. These are found to be Congruences of Rays to which we give the name *Dual Congruences*.

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* Geometrie der Dynamen. Teubner, 1903, pp. 195—202. Hereafter we shall refer to this work simply as „Dynamen“.

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On account of the simplicity of the equations and the corresponding simplification in the study of their properties, these congruences seem likely to play a significant part in the advancement of our knowledge of that geometry in which the straight line is taken as the element.

If we keep in mind the role that the Conic Section has played in Plane Geometry even when considered from the algebraic side alone, it seems very desirable that the corresponding development should be carried out for the Dual Congruences. This point is somewhat more strongly emphasized when we find that there exist dual linear transformations (dual collineations) leading to dual relations which are closely analogous to the projective relations in the plane, and in which the *dual congruence of the second order* plays much the same role as the conic section in the plane. They are called dual projective relations, and the corresponding geometry is called dual projective geometry. We are thus led very naturally to the problem of the following investigation: namely, the *classification and the determination of properties of configurations given by dual representations in which the variables enter to the second degree.*

After a preliminary discussion of dual quantities, coordinates and representations in Section I, we find in Sections II and III, a discussion of the general collineations, the special collineations due to motions, the reduction of the equations to canonical forms and their classification. In Section IV, after the introduction of a new kind of coordinates, the new equations for the previously given forms are found and also the equation corresponding to the general case of parametric representation; the idea of pencils of parallel rays is introduced. In Section V, is the discussion of the representations and configurations of the remaining cases of parametric representation, making use of coordinates of both kinds. In Section VI, after a discussion of quadratic equations in a single variable, there is given a development of the general properties of dual conical congruences as related to normal nets, reciprocal congruences, and theorems on the general distribution of the rays in these congruences. In Section VII, is found a geometrical description of the various classes of conical congruences as given in Sections II and III.

I. Preliminary Discussion.

§ 1. Dual Quantities and Dual Coordinates.

We will now turn to the consideration of these dual quantities.[*] Let

$$\mathfrak{x} = x + \xi\varepsilon, \qquad \text{where } \varepsilon^2 = 0.$$

Further, these quantities are otherwise to satisfy the laws of combination of the ordinary number-system.

If we have an integral function f of any number of dual quantities and perform the indicated operations, we can separate the resulting expression into two parts, one of which does not contain ε, and another which does contain it. The former is called the *scalar part* of the function and is represented by $S\mathfrak{f}$; the latter is called the *vector part* of the function and is represented by $V\mathfrak{f}$. Then if $\mathfrak{f} = 0$, we must have both $S\mathfrak{f} = 0$ and $V\mathfrak{f} = 0$.

If we consider that the ratio of the three dual quantities $X_1 : X_2 : X_3$ is unchanged when each of them is multiplied by the same dual quantity

$$\mathfrak{r} = r + \varrho\,\varepsilon, \qquad \text{where } r \neq 0,$$

we find that then there are exactly four independent scalar quantities, which we know is the number necessary for the complete determination of a straight line in the ordinary space of three dimensions.[†] This system has the particularly desirable property that there are no superfluous coordinates, i. e. that there are no additional fundamental relations to be satisfied.

Now let us write

$$X_i = \mathfrak{X}_{oi} + \mathfrak{X}_{jk}\,\varepsilon, \qquad (i = 1, 2, 3),$$

where i, j, k represent a cyclic arrangement of the numbers 1, 2, 3 in this same order[†]: also

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[*] Dynamen pp. 195—199.

[†] Dynamen p. 200.

$$(\mathfrak{X}/\mathfrak{Y}) = \mathfrak{X}_{01}\,\mathfrak{Y}_{01} + \mathfrak{X}_{02}\,\mathfrak{Y}_{02} + \mathfrak{X}_{03}\,\mathfrak{Y}_{03}\,,$$

and

$$(\mathfrak{X}\,\mathfrak{Y}) \equiv \mathfrak{X}_{01}\,\mathfrak{Y}_{23} + \mathfrak{X}_{02}\,\mathfrak{Y}_{31} + \mathfrak{X}_{03}\,\mathfrak{Y}_{12} + \mathfrak{X}_{23}\,\mathfrak{Y}_{01} + \mathfrak{X}_{31}\,\mathfrak{Y}_{02} + \mathfrak{X}_{12}\,\mathfrak{Y}_{03}. \quad ^{*}$$

If we multiply X_i $(i = 1, 2, 3)$ by the dual factor

$$(\mathfrak{X}/\mathfrak{X}) - {}^1/_2\,(\mathfrak{X}\,\mathfrak{X})\,\varepsilon$$

and rewrite, we find the following values for, $\mathfrak{X}_{oi}, \mathfrak{X}_{ik}$

$$\mathfrak{X}'_{oi}, = (\mathfrak{X}/\mathfrak{X})\,\mathfrak{X}_{oi}\,, \quad \mathfrak{X}'_{jk} = (\mathfrak{X}/\mathfrak{X})\,\mathfrak{X}_{jk} - {}^1/_2\,(\mathfrak{X}\,\mathfrak{X})\,\mathfrak{X}_{oi}\,, \quad (i = 1,\ 2,\ 3).$$

Further, these new values satisfy the fundamental relation for the Plücker Line-coordinates; namely,

$$(\mathfrak{X}\,\mathfrak{X}) = 0.$$

From this we see that the Ray-geometry includes the Plücker Line-geometry, and coincides with it in all cases where

$$(\mathfrak{X}/\mathfrak{X}) \neq 0.$$

We have the following two methods of selecting the rays of a congruence; namely, 1), that in which the quantities $\mathfrak{X}_{oi}$, $\mathfrak{X}_{jk}$, satisfy certain equations; 2), that in which these quantities are given as functions of two scalar parameters. In each of these two classes, we can immediately distinguish two subclasses as follows: in 1), according as the equations are or are not equivalent to those two which are obtained from a single equation in the three homogeneous dual coordinates X_i; and in a similar way in 2), according as the dual coordinates X_i can or can not each be represented as a function of a single dual parameter containing dual coefficients, as $X_i = \mathfrak{f}_i\,(\mathfrak{x})$, $(i = 1,\ 2,\ 3)$.

In the special types which we consider in this investigation, the functions involved are rational integral functions. These all belong to that type of functions of dual variables which corresponds to "analytic" functions in the ordinary complex variables. They are called "synectic" functions.† In these forms of representation, we say that the first method, in both cases 1) and 2), gives "synectic" representations, while the second method does not, although it may give analytic representations.

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* Dynamen, p. 129, (13); p. 138, (39).

† For the definiton of a synectic function of a dual variable see the Dynamen p. 199. Corresponding to the general definition, the above statement becomes: Synectic representations are obtained when the functions involved are synectic.

We reserve the name "synectic" Congruence however for such in which the dual coordinates of the rays are given as synectic functions of a single dual variable.

In all the following investigation, we shall use the German letters, in general, to represent dual quantities, as

$$\mathfrak{a} = a + \alpha\varepsilon ,$$

always with the exception of the dual coordinates themselves, which are represented as

$$X_i = \mathfrak{X}_{oi} + \mathfrak{X}_{jk} \,.\, \varepsilon$$

and in which $\mathfrak{X}_{oi}$, $\mathfrak{X}_{jk}$ have a significance analogous to the corresponding combinations in Plücker coordinates. Further, i, j, k when used together shall have the same signification as given heretofore, namely, they form a cyclic arrangement of the numbers 1, 2, 3 in this same order.

We shall also always consider that we have a tri-rectangular reference system in three dimensions for the homogeneous point-coordinates $x_0 : x_1 : x_2 : x_3$, where $x_0 = 0$ represents the plane at infinity, and where the Plücker coordinates of the line passing through the two points x and y are expressed as

$$\varrho\,\mathfrak{X}_{oi} = x_o\,y_i - x_i\,y_o\,,\quad \varrho\,\mathfrak{X}_{jk} = x_j\,y_k - x_k\,y_j\,,\ (i = 1,\ 2,\ 3).$$

§ 2. Quadratic Synectic Representations.

We will now return to the two classes of synectic representation mentioned above, limiting ourselves however to those of the second degree.

We have

$$1)\qquad \mathfrak{S} \equiv \mathfrak{a}_{11}\,X_1{}^2 + \mathfrak{a}_{22}\,X_2{}^2 + \mathfrak{a}_{33}\,X_3{}^2 + 2\,\mathfrak{a}_{23}\,X_2\,X_3 + 2\,\mathfrak{a}_{31}\,X_3\,X_1 + 2\,\mathfrak{a}_{12}\,X_1\,X_2 = 0$$

as the general form of the equation to be satisfied by the coordinates of the rays in case 1).

We will call the discriminant of this equation, that function of its dual coefficients $\mathfrak{a}_{ik}$ which is formed in the same way as the discriminant in the general ternary quadratic form in scalar quantities, and will represent it by

$$\varDelta \equiv (\mathfrak{a}_{11}\, \mathfrak{a}_{22}\, \mathfrak{a}_{33}).$$

Further we define the "general" case as that for which

$$S\varDelta \neq 0.$$

For the synectic parametric representation, the dual coordinates are given by

$$2) \qquad X_i = \mathfrak{M}_{i11}\, \mathfrak{t}_1^2 + 2\, \mathfrak{M}_{i12}\, \mathfrak{t}_1\, \mathfrak{t}_2 + \mathfrak{M}_{i22}\, \mathfrak{t}_2^2, \ (i = 1,\ 2,\ 3),$$

in which we will suppose that there is no dual factor common to all the coefficients $\mathfrak{M}$. Writing I for the determinant of the the coefficients

$$I \equiv (\mathfrak{M}_{111}\, \mathfrak{M}_{212}\, \mathfrak{M}_{322}),$$

we will define the "general" case under this class as that for which

$$SI \neq 0.$$

By methods of elimination analogous to those used in the corresponding discussion in the theory of the Conic Section, we can always find an equation of the second degree in X_1, X_2, X_3 which will be satisfied by the ray-coordinates as given by 2).

Since this system of dual quantities is such that the combination of the scalar parts of the elements of any algebraic combination involving only addition, subtraction and multiplication gives the scalar part of the resulting expression, we can further conclude from the theory of Conic Sections, that the resulting equation of the second degree is such that $S\varDelta \neq 0$ if $SI \neq 0$; and also, that if $SI = 0$, then all the first minors of $S\varDelta$ are zero and hence that $\varDelta = 0$. From this it is readily seen that the congruences excluded by the condition $S\varDelta \neq 0$ in 1) do not all belong to the set excluded by the condition $SI \neq 0$ in 2); namely, those satisfying the conditions $S\varDelta = 0$, $\varDelta \neq 0$, and $\varDelta = 0$, but all first minors of

$S\Delta$ not zero. On the other hand, those congruences whose equations fulfil the condition $SI = 0$ are included among those for which $S\Delta = 0$.

II. Transformation of the Equations by dual (radial) Collineations. Canonical Forms.

§ 3. Collineations. Canonical Forms under dual Collineations

Starting from the general equation of the second degree, we will separate into the scalar and vector parts. The former we will represent by S and the latter by $\Sigma\varepsilon$.

Then we write

1) $$\mathfrak{S} \equiv S + \Sigma\varepsilon$$

where

2) $$S \equiv a_{11}\mathfrak{X}_{01}^2 + a_{22}\mathfrak{X}_{02}^2 + a_{33}\mathfrak{X}_{03}^2 + 2\,a_{23}\mathfrak{X}_{02}\mathfrak{X}_{03} + 2\,a_{31}\mathfrak{X}_{03}\mathfrak{X}_{01} + 2\,a_{12}\mathfrak{X}_{01}\mathfrak{X}_{02}$$

Further defining S_i and Ω as follows

3) $$S_i \equiv \frac{1}{2}\,\frac{\delta S}{\delta \mathfrak{X}_{oi}}\,, \qquad (i = 1,\ 2,\ 3)$$

4) $$\Omega \equiv {}^1/_2\,(a_{11}\mathfrak{X}_{01}^2 + a_{22}\mathfrak{X}_{02}^2 + a_{33}\mathfrak{X}_{03}^2 + 2\,a_{23}\mathfrak{X}_{02}\mathfrak{X}_{03} + 2\,a_{31}\mathfrak{X}_{03}\mathfrak{X}_{01} + 2\,a_{12}\mathfrak{X}_{01}\mathfrak{X}_{02}),$$

we have

5) $$\Sigma \equiv 2\,(S_1\mathfrak{X}_{23} + S_2\mathfrak{X}_{31} + S_3\mathfrak{X}_{12} + \Omega).$$

Taking the idea of equivalence as used in the theory of transformations to mean that two equations (also the configurations defined by them) are equivalent under any group of transformations, if one can be obtained from the other by means of transformations of the group, we will turn to the determination of the different equations under the dual and radial collineations. These collineations are defined as follows.

For the dual collineations, we have

$$6) \qquad X_i = \mathfrak{m}_{i1} X_1' + \mathfrak{m}_{i2} X_2' + \mathfrak{m}_{i3} X_3', \qquad (i = 1, 2, 3);$$ [*]

for the radial collineations,

$$7) \qquad \begin{aligned} \mathfrak{X}_{0i} &= m_{i1}\, \mathfrak{X}'_{01} + m_{i2}\, \mathfrak{X}'_{02} + m_{i3}\, \mathfrak{X}'_{03} \\ \mathfrak{X}_{jk} &= \mu_{i1}\, \mathfrak{X}'_{01} + \mu_{i2}\, \mathfrak{X}'_{02} + \mu_{i3}\, \mathfrak{X}'_{03} \\ &\quad + l\,(m_{i1}\, \mathfrak{X}'_{23} + m_{i2}\, \mathfrak{X}'_{31} + m_{i3}\, \mathfrak{X}'_{12}), \end{aligned}$$

where $l\,(m_{11}\, m_{22}\, m_{33}) \neq 0, \qquad (i = 1, 2, 3).$ [**]

We see that the dual collineation is that special case of the radial collineation for which $l = 1$.

Transformations with general coefficients.

Using the results of the theory of transformation of quadratic ternary forms as obtained in the theory of the Conic Section, we see immediately that a dual collineation

$$8) \qquad X_i = m_{i1} X_1' + m_{i2} X_2' + m_{i3} X_3', \qquad (i = 1, 2, 3)$$

$(m_{11}\, m_{22}\, m_{33}) \neq 0$, where m_{ik} are all scalar,
can be found such that the new S after dropping the primes, will reduce to one of the three following forms:

$$9) \quad \begin{cases} a) \ \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2, & \text{if } S\varDelta \neq 0; \\ b) \ \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2, & \text{if } S\varDelta = 0, \text{ but } \textit{not} \text{ all its first minors zero;} \\ c) \ \mathfrak{X}_{01}^2, & \text{if } S\varDelta = 0 \text{ and all its first minors zero.} \end{cases}$$

For the corresponding cases, we have the following expressions for the new Σ respectively:

$$10) \quad \begin{cases} a) \ 2\,[\,(\mathfrak{X}_{01}\, \mathfrak{X}_{23} + \mathfrak{X}_{02}\, \mathfrak{X}_{31} + \mathfrak{X}_{03}\, \mathfrak{X}_{12}) + \Omega], \\ b) \ 2\,[\,(\mathfrak{X}_{01}\, \mathfrak{X}_{23} + \mathfrak{X}_{02}\, \mathfrak{X}_{31}) + \Omega], \\ c) \ 2\,[\,\mathfrak{X}_{01}\, \mathfrak{X}_{23} + \Omega]. \end{cases}$$

The equations can be still further simplified by applying a dual Translation [†], which is written in general form as,

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[*] Dynamen p. 224.

[**] Dynamen p. 237.

[†] Dynamen p. 235.

$$11)\quad \left\{\begin{aligned} X_1 &= (1+\mu_{11}\,\varepsilon)\,X_1' + \varepsilon\,(\mu_{12}\,\mathfrak{X}'_{02} + \mu_{13}\,\mathfrak{X}'_{03})\,, \\ X_2 &= \mu_{21}\,\varepsilon\,\mathfrak{X}'_{01} + (1+\mu_{22}\,\varepsilon)\,X_2' + \mu_{23}\,\varepsilon\,\mathfrak{X}'_{03}\,, \\ X_3 &= \varepsilon\,(\mu_{31}\,\mathfrak{X}'_{01} + \mu_{32}\,\mathfrak{X}'_{02}) + (1+\mu_{33}\,\varepsilon)\,X_3'\,, \end{aligned}\right.$$

where μ_{ik} are all scalar quantities. S is unchanged by such a transformation and the changes in Σ will now be discussed for each of the three cases *a), b), c)* separately.

a.

Applying this transformation 11), the expression 10 *a*) becomes

$$12\,a)\quad 2\,(\mathfrak{X}_{01}\,\mathfrak{X}_{23} + \mathfrak{X}_{02}\,\mathfrak{X}_{31} + \mathfrak{X}_{03}\,\mathfrak{X}_{12}) + (a_{11} + 2\mu_{11})\,\mathfrak{X}_{01}^2 + (a_{22} + 2\,\mu_{22})\,\mathfrak{X}_{02}^2 + (a_{33} + 2\,\mu_{33})\,\mathfrak{X}_{03}^2 + 2\,(a_{23} + \mu_{23} + \mu_{32})\,\mathfrak{X}_{02}\mathfrak{X}_{03} + 2\,(a_{31} + \mu_{31} + \mu_{13})\,\mathfrak{X}_{03}\,\mathfrak{X}_{01} + 2\,(a_{12} + \mu_{12} + \mu_{21})\mathfrak{X}_{01}\,\mathfrak{X}_{02}.$$

The coefficients of all the terms $\mathfrak{X}_{oi}\,\mathfrak{X}_{oj}$ $(i, j = 1, 2, 3)$ can be made zero and still leave three of the quantities μ_{ik} arbitrary. Hence;

When $S\varDelta \neq 0$, the equation of the second degree in dual coordinates can be transformed by dual collineations into one whose form is

$$13\,a)\qquad X_1^2 + X_2^2 + X_3^2 = 0$$

b.

By the same transformation 11), the expression 10 *b*) becomes

$$12\,b)\quad 2\,(\mathfrak{X}_{01}\,\mathfrak{X}_{23} + \mathfrak{X}_{02}\,\mathfrak{X}_{31}) + (a_{11} + 2\,\mu_{11})\,\mathfrak{X}_{01}^2 + (a_{22} + 2\,\mu_{22})\,\mathfrak{X}_{02}^2 + a_{33}\,\mathfrak{X}_{03}^2 + 2\,(a_{23} + \mu_{23})\,\mathfrak{X}_{02}\,\mathfrak{X}_{03} + 2\,(a_{31} + \mu_{13})\,\mathfrak{X}_{03}\,\mathfrak{X}_{01} + 2\,(a_{12} + \mu_{12} + \mu_{21})\,\mathfrak{X}_{01}\,\mathfrak{X}_{02}.$$

In this case, all the coefficients of $\mathfrak{X}_{oi}\,\mathfrak{X}_{oj}$ can be made zero, *except that of* $\mathfrak{X}_{03}{}^2$, and still leave four coefficients μ_{ik} arbitrary.

Here we have two cases according as a_{33} is zero or not, and our equation becomes

$$14\,b)\qquad \left\{\begin{aligned} &\alpha)\quad X_1^2 + X_2^2 + \varepsilon\,a_{33}\,X_3^2 = 0, \quad \text{or} \\ &\beta)\quad X_1^2 + X_2^2 = 0. \end{aligned}\right.$$

In the first of these, we can still further simplify by applying the transformation,

$$X_1 = X_1', \qquad X_2 = X'_2, \qquad \sqrt{a_{33}}\, X_3 = X_3'.$$

We then obtain the following result:

When $S\varDelta = 0$ but all its first minors not zero, the equation of the congruence can be reduced by dual collineations to the form

$$13\,\text{b)} \quad \begin{cases} \alpha) \quad X_1^2 + X_2^2 + \varepsilon X_3^2 = 0, & \text{if } \varDelta \neq 0, \text{ or} \\ \beta) \quad X_1^2 + X_2^2 = 0, & \text{if } \varDelta = 0. \end{cases}$$

c.

The expression 10 *c*) by the same transformation becomes

$$12\,\text{c)} \quad \begin{aligned} &2\mathfrak{X}_{01}\mathfrak{X}_{23} + (a_{11} + 2\mu_{11})\mathfrak{X}_{01}^2 + a_{22}\mathfrak{X}_{02}^2 + a_{33}\mathfrak{X}_{03}^2 \\ &+ 2a_{23}\mathfrak{X}_{02}\mathfrak{X}_{03} + 2(a_{31} + \mu_{13})\mathfrak{X}_{03}\mathfrak{X}_{01} + 2(a_{12} + \mu_{12})\mathfrak{X}_{01}\mathfrak{X}_{02}. \end{aligned}$$

Here the coefficients of $\mathfrak{X}_{02}^2$, $\mathfrak{X}_{03}^2$, $\mathfrak{X}_{02}\,\mathfrak{X}_{03}$ can *not* be changed, but the coefficients of $\mathfrak{X}_{01}^2$, $\mathfrak{X}_{03}\,\mathfrak{X}_{01}$, $\mathfrak{X}_{01}\,\mathfrak{X}_{02}$ can be made zero and leave six of the coefficients μ_{ik} arbitrary. The equation then takes one of three forms according to the different relations between a_{22}, a_{33}, a_{23}, as follows:

$$14\,\text{c)} \quad \begin{cases} \alpha) \; X_1^2 + \varepsilon(\beta_2 X_2 + \beta_3 X_3)(\gamma_2 X_2 + \gamma_3 X_3) = 0, & \text{if } a^2_{23} - a_{22}a_{33} \neq 0, \\ \beta) \; X_1^2 + \varepsilon(\beta_2 X_2 + \beta_3 X_3)^2 = 0, & \text{if } a^2_{23} - a_{22}a_{33} = 0, \\ \gamma) \; X_1^2 = 0, & \text{if } a_{22} = a_{23} = a_{33} = 0. \end{cases}$$

In α), by the transformation given by the equations

$$X_1 = X_1', \quad \beta_2 X_2 + \beta_3 X_3 = X_1' + iX_3', \quad \gamma_2 X_2 + \gamma_3 X_3 = X_2' - iX_3',$$

and, in β), by the transformation given by

$$X_1 = X_1', \quad \beta_2 X_2 + \beta_3 X_3 = X_2',$$

the equations are further simplified, and we obtain the following result:

If $S\varDelta = 0$ and all its first minors are zero, the equation of the congruence can be reduced by dual collineations to one of the three forms:

$$12\,\text{c)} \quad \begin{cases} \alpha) \quad X_1^2 + \varepsilon(X_2^2 + X_3^2) = 0, \\ \beta) \quad X_1^2 + \varepsilon X_2^2 = 0, \\ \gamma) \quad X_1^2 = 0. \end{cases}$$

§ 4. Canonical Forms under radial Collineations.

From the equations 7) defining the radial collineations, we see that we obtain exactly the same transformation if we first apply the dual collineation obtained by making $l = 1$, and then the transformation

15) $$\mathfrak{X}_{oi} = \mathfrak{X}'_{oi}, \; \mathfrak{X}_{jk} = l\mathfrak{X}'_{jk} \qquad (i = 1, 2, 3),$$

which, in the Plücker coordinates results from the application of a perspective similarity point-transformation with the origin as center. The result of applying a radial collineation to an equation synectic in X_i differs from that for the corresponding dual collineation only in that the coefficients of $\mathfrak{X}_{jk}$ in Σ are multiplied by l. The same congruence would be given by the synectic equation,

16) $$S + 2\left(S_1\mathfrak{X}_{23} + S_2\mathfrak{X}_{31} + S_3\mathfrak{X}_{12} + \frac{1}{l}\Omega\right)\varepsilon = 0,$$

where the equation with $l = 1$ is the result of the application of the corresponding dual collineation; i. e. the result of applying 15) is essentially a division of the coefficients a in Ω by l. This result is true for any integral synectic equation since Σ contains the quantities $\mathfrak{X}_{jk}$ to the first power only, on account of the fundamental property of the dual quantities, namely $\varepsilon^2 = 0$. Analytically, the equation 16) could be obtained by dividing the expressions given for $\mathfrak{X}_{jk}$ in 7) by l. In this case, the geometrical significance as given in 15) would be hidden, and some of the geometrical properties of the configuration might be overlooked.

The application of such a radial transformation does not affect the process of the elimination of terms by the application of 11), nor does it change the possibilities as given in 14b) and 14c). Therefore the same resulting forms are obtainable by applying radial collineations.

Rearranging and summarizing our results, we can say:

In radial- (dual-) projective geometry, there are not more than six essentially different configurations represented by properly synectic equations of the second degree. These can always be reduced to the following canonical forms:

17)

I. $\Delta \neq 0.$

A) $X_1{}^2 + X_2{}^2 + X_3{}^2 = 0\,,$ $S\Delta \neq 0;$

B) $X_1{}^2 + X_2{}^2 + \varepsilon X_3{}^2 = 0\,,$ $S\Delta = 0\,.$

II. $\Delta = 0.$

A) $X_1{}^2 + X_2{}^2 = 0;$ *not* all first minors of $S\Delta$ zero.

B) All first minors of $S\Delta$ zero, but *not* all the elements of $S\Delta$ zero.

- *α)* $X_1{}^2 + \varepsilon(X_1{}^2 + X_2{}^2) = 0\,,$
- *β)* $X_1{}^2 + \varepsilon X_2{}^2 = 0\,,$
- *γ)* $X_1{}^2 = 0\,.$

The excluded case, namely that in which all the elements of $S\Delta$ are zero, does not give us a congruence but a quadratic complex whose general equation is

$$2\,\Omega = 0\,.$$

§ 5. Canonical Forms under radial Collineations with real Coefficients.

Limiting ourselves to synectic equations and radial collineations which have real coefficients only, we see from the preceding discussions of the cases a), b), c) § 3, that,

In real radial-(dual-)projective geometry, the essentially different proper synectic equations of the second degree with real coefficients can be reduced to the following canonical forms:

18)

I. $\Delta \neq 0.$

A) $X_1{}^2 \pm X_2{}^2 \pm X_3{}^2 = 0,$ $S\Delta \neq 0;$

B) $X_1{}^2 \pm X_2{}^2 \pm \varepsilon X_3{}^2 = 0,$ $S\Delta = 0.$

II $\Delta = 0$

A) $X_1{}^2 \pm X_2{}^2 = 0,$ *not* all first minors of $S\Delta$ zero.

B) All first minors of $S\Delta$ zero, but *not* all the elements of $S\Delta$ zero.

- *α)* $X_1{}^2 \pm \varepsilon(X_2{}^2 \pm X_3{}^2) = 0\,,$
- *β)* $X_1{}^2 \pm \varepsilon X_2{}^2 = 0\,,$
- *γ)* $X_1{}^2 = 0\,.$

In the above canonical forms, the different double signs are independent of one another.

III. Transformations of Equations due to Motions. Canonical Forms.

§ 6. Collineations for Motions and corresponding general Canonical Form.

We will now consider the transformations of the ray coordinates due to translations and rotations of the tri-rectangular reference system and also certain standard forms to which the synectic equations can be reduced by them when the coefficients are real. We limit ourselves to this class of cases because of their especial interest.

The transformation of rectangular point-coordinates which results from a rotation of the axes about the origin, can be written

$$x_0 : x_1 : x_2 : x_3 = l_{00} x_0' : l_{11} x_1' + l_{12} x_2' + l_{13} x_3' :$$
$$l_{21} x_1' + l_{22} x_2' + l_{23} x_3' : l_{31} x_1' + l_{32} x_2' + l_{33} x_3',$$

where $l_{ij} : l_{00}$ $(ij = 1, 2, 3)$ are the direction-cosines for the one system relative to the other, and where primed letters represent the new coordinates. The corresponding transformation in ray-coordinates is

1) $$X_1 : X_2 : X_3 = l_{11} X_1' + l_{12} X_2' + l_{13} X_3' :$$
$$l_{21} X_1' + l_{22} X_2' + l_{23} X_3' : l_{31} X_1' + l_{32} X_2' + l_{33} X_3'. \quad *$$

The transformation in point coordinates for the translation of the origin to the point $a_0 : a_1 : a_2 : a_3$ is

$$x_0 : x_1 : x_2 : x_3 = a_0 x_0' : a_0 x_1' + a_1 x_0' : a_0 x_2' + a_2 x_0' : a_0 x_3' + a_3 x_0';$$

and the corresponding transformation in ray-coordinates is

2) $$X_1 : X_2 : X_3 = a_0 X_1' + \varepsilon (a_2 \mathfrak{X}'_{03} - a_3 \mathfrak{X}'_{02}) :$$
$$a_0 X_2' + \varepsilon (a_3 \mathfrak{X}'_{01} - a_1 \mathfrak{X}'_{03}) : a_0 X_3' + \varepsilon (a_1 \mathfrak{X}'_{02} - a_2 \mathfrak{X}'_{01}).$$

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*) cf. Clebsch-Lindemann: Vorlesungen über Geometrie. II. pp, 78-79.

From the results of the transformation of ternary quadratic forms as used in the analytical geometry of three dimensions for the purpose of transforming the equation of a quadric surface to its principal axes as reference system, we can conclude that an equation of the form $S = 0$ can always be reduced to the form

$$3)\qquad b_{11}\mathfrak{X}_{01}^2 + b_{22}\mathfrak{X}_{02}^2 + b_{33}\mathfrak{X}_{03}^2 = 0$$

by a real transformation 1) if the coefficients a_{ik} of S are real. *

Our synectic quadratic equation then becomes

$$4)\qquad (b_{11} + \varepsilon\beta_{11}) X_1^2 + (b_{22} + \varepsilon\beta_{22}) X_2^2 + (b_{33} + \varepsilon\beta_{33}) X_3^2 \\ + 2\varepsilon(\beta_{23}\mathfrak{X}_{02}\mathfrak{X}_{03} + \beta_{31}\mathfrak{X}_{03}\mathfrak{X}_{01} + \beta_{12}\mathfrak{X}_{01}\mathfrak{X}_{02}) = 0;$$

for which the coefficient of the vector part is

$$5)\qquad \beta_{11}\mathfrak{X}_{01}^2 + \beta_{22}\mathfrak{X}_{02}^2 + \beta_{33}\mathfrak{X}_{03}^2 + 2\beta_{23}\mathfrak{X}_{02}\mathfrak{X}_{03} + 2\beta_{31}\mathfrak{X}_{03}\mathfrak{X}_{01} \\ + 2\beta_{12}\mathfrak{X}_{01}\mathfrak{X}_{02} + 2b_{11}\mathfrak{X}_{01}\mathfrak{X}_{23} + 2b_{22}\mathfrak{X}_{02}\mathfrak{X}_{31} + 2b_{33}\mathfrak{X}_{03}\mathfrak{X}_{12}.$$

Now applying the transformation 2) for a translation of the origin, 4) gives

$$3')\qquad a_0^2(b_{11}\mathfrak{X}_{01}^2 + b_{22}\mathfrak{X}_{02}^2 + b_{33}\mathfrak{X}_{03}^2) = 0.$$

$$5')\quad \begin{cases} a_0 \{ a_0 [\beta_{11}\mathfrak{X}_{01}^2 + \beta_{22}\mathfrak{X}_{02}^2 + \beta_{33}\mathfrak{X}_{03}^2 + 2\beta_{23}\mathfrak{X}_{02}\mathfrak{X}_{03} + 2\beta_{31}\mathfrak{X}_{03}\mathfrak{X}_{01} \\ \quad + 2\beta_{12}\mathfrak{X}_{01}\mathfrak{X}_{02} + 2b_{11}\mathfrak{X}_{01}\mathfrak{X}_{23} + 2b_{22}\mathfrak{X}_{02}\mathfrak{X}_{31} + 2b_{33}\mathfrak{X}_{03}\mathfrak{X}_{12}] \\ \quad - [2a_1(b_{22} - b_{33})\mathfrak{X}_{02}\mathfrak{X}_{03} + 2a_2(b_{33} - b_{11})\mathfrak{X}_{03}\mathfrak{X}_{01} \\ \qquad + 2a_3(b_{11} - b_{22})\mathfrak{X}_{01}\mathfrak{X}_{02}]\} = 0. \end{cases}$$

The coefficients of the product terms are the only ones that are essentially changed, and we see that these can be made zero by a proper choice of the coordinates $a_0 : a_1 : a_2 : a_3$, when

$$6)\qquad (b_{22} - b_{33})(b_{33} - b_{11})(b_{11} - b_{22}) \neq 0.$$

Then we have

$$7)\qquad a_0 : a_1 : a_2 : a_3 = 1 : \frac{\beta_{23}}{b_{22} - b_{33}} : \frac{\beta_{31}}{b_{33} - b_{11}} : \frac{\beta_{12}}{b_{11} - b_{22}}.$$

That is, under this condition 6), the equation 4) can be reduced to the form

$$8a)\qquad \mathfrak{a}_{11} X_1^2 + \mathfrak{a}_{22} X_2^2 + \mathfrak{a}_{33} X_3^2 = 0.$$

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*) Clebsch-Lindemann: Geometrie II., pp. 165—170.
Salmon-Fiedler: Geometrie des Raumes I.

If the expression in 6) vanishes, we have two types according as $b_{22} = b_{33}$ or $b_{11} = b_{22} = b_{33}$. In these two cases the equations can be written in the following forms respectively:

$$8\text{b})\quad a_{11} X_1^2 + a_{22}(X_2^2 + X_3^2) + \varepsilon(a_{22} X_2^2 + 2\, a_{23} X_2 X_3 + a_{33} X_3^2) = 0$$

$$8\text{c})\quad a_{11}(X_1^2 + X_2^2 + X_3^2) + \varepsilon(a_{11} X_1^2 + a_{22} X_2^2 + a_{33} X_3^2$$
$$+ 2\, a_{23} X_2 X_3 + 2\, a_{31} X_3 X_1 + 2\, a_{12} X_1 X_2) = 0\ .$$

In the former case the coordinate a_1 is entirely arbitrary, and in the latter a_1, a_2, a_3, and entirely without effect so far as changing the coefficients of the equation is concerned, as we can easily see by a consideration of equation 5').

Now for 8 b) we can find a real rotation expressed by equations of the form

$$X_1 = X_1',\ X_2 = l_{22} X_2' + l_{23} X_3',\ X_3 = l_{32} X_2' + l_{33} X_3'$$

such that $a_{22} X_2^2 + 2\, a_{23} X_2 X_3 + a_{33} X_3^2$ will take the form $a'_{22} X_2^2 + a'_{33} X_3^2$ and $X_2^2 + X_2^2$ still remain unchanged.

Similarly, for 8 c) we can find real rotations such that

$$a_{11} X_1^2 + a_{22} X_2^2 + a_{33} X_3^2 + 2\, a_{23} X_2 X_3 + 2\, a_{31} X_3 X_1 + 2\, a_{12} X_1 X_2$$

becomes $a'_{11} X_1^2 + a'_{22} X_2^2 + a'_{33} X_3^2$ while $X_1^2 + X_2^2 + X_3^2$ will remain unchanged. Therefore 8 b) and 8 c) can be reduced to the form 8 a). Summarizing, we have the result:

A synectic equation of the second degree in ray-coordinates which has only real coefficients can always be reduced by real motions of the reference-system, to the form

$$9)\qquad a_{11} X_1^2 + a_{22} X_2^2 + a_{33} X_3^2 = 0\ .$$

§ 7. Different special Canonical Forms.

The points which come into the consideration of equations 8 a), b), c) through 5') and 7) might be readily considered as centers of the corresponding congruences, in case we limited ourselves to such as have equivalents in the Plücker line-continuum. Making use of this idea, we will consider the question again, this time using equation 9) for greater clearness.

When $S\,(\alpha_{22}-\alpha_{33})\,(\alpha_{33}-\alpha_{11})\,(\alpha_{11}-\alpha_{22}) \neq 0$, we find that the center is entirely determined and that its coordinates are $1:0:0:0$.

If $S\,(\alpha_{22}-\alpha_{33})\,(\alpha_{33}-\alpha_{11})\,(\alpha_{11}-\alpha_{22}) = 0$, and but one factor of the product is zero, as $S\,(\alpha_{22}-\alpha_{33})$, then one of the coordinates becomes indeterminate; in this case a_1, but the others are all definite. Here, then, we may say that there is a line of centers.

The remaining possibility is where $S\alpha_{11} = S\alpha_{22} = S\alpha_{33}$. In this case, the center is entirely indeterminate, since this is true of the three coordinates a_1, a_2, a_3.

This idea that the configuration represented has a single definite center, or any point of a definite line as center, or an entirely indeterminate center (or every point in space) gives a criterion for a different division into classes which can be carried out simultaneously with the one of § 4. These classes are three in number and will be designated by a), b), c); in which we have $S\,(\alpha_{11}-\alpha_{22})\,(\alpha_{22}-\alpha_{33})\,(\alpha_{33}-\alpha_{11}) \neq 0$ for class a, $S\,(\alpha_{11}-\alpha_{22})\,(\alpha_{22}-\alpha_{33})\,(\alpha_{33}-\alpha_{11}) = 0$ for class b, and $S\alpha_{11} = S\alpha_{22} = S\alpha_{33}$ for class c.

In writing the equations in the following classification, the coefficients have been selected in such a way as to obtain the greatest possible symmetry, and in most cases unity has been used for the equal coefficients a_{ii}. We then state our result as follows:

A synectic equation of the second degree in ray-coordinates with real coefficiants can always be reduced by real motions to one of the following canonical forms:

$$
10)\left\{\begin{array}{ll}
 & \qquad I. \qquad\qquad \Delta \neq 0 \,. \\
 & \qquad A. \qquad\qquad S\Delta \neq 0 \,. \\
a) & (a_{11}+\alpha_{11}\,\varepsilon)\,X_1^2+(a_{22}+\alpha_{22}\,\varepsilon)\,X_2^2+(a_{33}+\alpha_{33}\,\varepsilon)\,X_3^2 = 0 \,, \\
b) & X_1^2+a_{22}\,(X_2^2+X_3^2)+\varepsilon(\alpha_{22}\,X_2^2+\alpha_{33}\,X_3^2) = 0 \,, \\
c) & X_1^2+X_2^2+X_3^2+\varepsilon\,(\alpha_{11}\,X_1^2+\alpha_{22}\,X_2^2+\alpha_{33}\,X_3^2) = 0 \,. \\
 & \qquad B. \qquad\qquad S\Delta = 0 \,. \\
a) & (a_{11}+\alpha_{11}\,\varepsilon)\,X_1^2+(a_{22}+\alpha_{22}\,\varepsilon)\,X_2^2+\varepsilon\,\alpha_{33}\,X_3^2 = 0 \,. \\
b) & X_1^2+X_2^2+\varepsilon\,(\alpha_{22}\,X_2^2+\alpha_{33}\,X_3^2) = 0 \,.
\end{array}\right.
$$

10)

II. $\Delta = 0.$

A. First minors of $S\Delta$ *not* all zero.

a) $$(a_{11} + a_{11}\varepsilon) X_1^2 + (a_{22} + a_{22}\varepsilon) X_2^2 = 0 \,,$$

b) $$X_1^2 + X_2^2 + \varepsilon (a_{11} X_1^2 + a_{22} X_2^2) = 0 \,.$$

B. First minors of $S\Delta$ all zero.

b

α) $$X_1^2 + \varepsilon (a_{22} X_2^2 + a_{33} X_3^2) = 0 \,,$$

β) $$X_1^2 + \varepsilon a_{22} X_2^2 = 0 \,,$$

γ) $$X_1^2 = 0 \,.$$

IV. Coordinates of the Second Kind. Equations of the Configurations for the General Cases. Parallel Pencils.

§ 8. Coordinates of the Second Kind.

Although the dual coordinates in the ray-geometry are so closely analogous to the homogeneous coordinates in a plane and leave nothing to be desired in regard to the finite domain, nevertheless we do not obtain a closed continuum of any type corresponding to that of the projective geometry, in so much as we do not obtain definite rays as limits when we pass to the region at infinity, i. e. where $\mathfrak{X}_{01} = \mathfrak{X}_{02} = \mathfrak{X}_{03} = 0$.

In order to overcome this difficulty the following system of coordinates was introduced.*

1) $$\mathfrak{X}_i = \mathfrak{X}_{oi}\,; \qquad \mathfrak{X}_{ii} = \begin{vmatrix} \mathfrak{X}_{oj} \mathfrak{X}_{ki} \\ \mathfrak{X}_{ok} \mathfrak{X}_{ij} \end{vmatrix}, \qquad (i = 1,\ 2,\ 3)\ .$$

They are called coordinates of the second kind, or coordinates of the first natural system and they satisfy the following relation identically:

2) $$\mathfrak{X}_1 \mathfrak{X}_{11} + \mathfrak{X}_2 \mathfrak{X}_{22} + \mathfrak{X}_3 \mathfrak{X}_{33} = 0\ .$$

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* Dynamen pp. 258—261.

Further, these coordinates are homogeneous in a new sense: namely, in that the same ray is represented if we multiply $\mathfrak{X}_i$ by ϱ, and at the same time $\mathfrak{X}_{ii}$ by ϱ^2 for $i = 1, 2, 3$.

By this change a new continuum is introduced so related to the original one that to every ray of the one there corresponds one and but one ray of the other, *except when*

$$3) \qquad \mathfrak{X}_{01} = \mathfrak{X}_{02} = \mathfrak{X}_{03} = 0, \text{ i. e. } \mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}_3 = 0.$$

In the first natural continuum, these special rays are called *Point Rays,* and they make up the totality of rays which are situated at infinity. All other rays are called *Proper Rays.*

From this we see that we can discuss the configurations in either continuum by means of the corresponding equation in the other, *except* for the rays satisfying 3), and these must be considered separately.

§ 9. Equations for Case 1), § 2.

We will now find and discuss the system of equations in coordinates of the second kind which must be satisfied: 1) for the dual equation $\mathfrak{S} = 0$; and 2), for the general case of the dual parametric representation of the second degree. In Section V, we will consider those for the case $SI = 0$. All the following discussions will deal with the configurations obtained in the new continuum.

Making use of the symbols defined by 2), 3), 4), § 3, we have immediately

$$4) \qquad S = 0.$$

Multiply $^1/_2\, \Sigma = 0$ by $\mathfrak{X}_{oi}$, subtract $\mathfrak{X}_{jk}(\mathfrak{X}_{01} S_1 + \mathfrak{X}_{02} S_2 + \mathfrak{X}_{03} S_3) = 0$ and then substitute in terms of the new coordinates. Making i successively equal 1, 2, 3, we obtain the following equations:

$$5) \left\{ \begin{aligned} -S_3 \mathfrak{X}_{22} + S_2 \mathfrak{X}_{33} + \mathfrak{X}_1 \Omega &= 0, \\ S_3 \mathfrak{X}_{11} \qquad\quad - S_1 \mathfrak{X}_{33} + \mathfrak{X}_2 \Omega &= 0, \\ -S_2 \mathfrak{X}_{11} + S_1 \mathfrak{X}_{22} \qquad\quad + \mathfrak{X}_3 \Omega &= 0. \end{aligned} \right.$$

By further examination of this system of equations 2), 4), 5), we find that, considering the system made up of 2) and 5), the determinants of the third order of the matrix of their coefficients

$$\begin{Vmatrix} 0 & -S_3 & S_2 & \mathfrak{X}_1 \\ S_3 & 0 & -S_1 & \mathfrak{X}_2 \\ -S_2 & S_1 & 0 & \mathfrak{X}_3 \\ \mathfrak{X}_1 & \mathfrak{X}_2 & \mathfrak{X}_3 & 0 \end{Vmatrix}$$

are all congruent to zero (mod S); and therefore for the rays which are to be considered, these determinants must all be zero on account of 4), and hence the equations 2), 5) are not independent for any rays of the configuration. The determinants of the second order all vanish only if $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}_3 = 0$, and in this case the whole system 2), 4), 5) is satisfied.

Since the point-rays all satisfy the system of equations 2), 4), 5), it is unsatisfactory for the determination of a closed configuration. We will now obtain a new equation which is particularly valuable in this connection.

We rewrite equations 5), arranging according to $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$, as follows:

$$5') \quad \begin{cases} (\Omega - a_{31}\mathfrak{X}_{22} - a_{12}\mathfrak{X}_{33})\mathfrak{X}_1 + (-a_{23}\mathfrak{X}_{22} + a_{22}\mathfrak{X}_{33})\mathfrak{X}_2 \\ \qquad + (-a_{33}\mathfrak{X}_{22} + a_{23}\mathfrak{X}_{33})\mathfrak{X}_3 = 0, \\ (a_{31}\mathfrak{X}_{11} - a_{11}\mathfrak{X}_{33})\mathfrak{X}_1 + (\Omega + a_{23}\mathfrak{X}_{11} - a_{12}\mathfrak{X}_{33})\mathfrak{X}_2 \\ \qquad + (a_{33}\mathfrak{X}_{11} - a_{31}\mathfrak{X}_{33})\mathfrak{X}_3 = 0, \\ (-a_{12}\mathfrak{X}_{11} + a_{11}\mathfrak{X}_{22})\mathfrak{X}_1 + (-a_{22}\mathfrak{X}_{11} + a_{12}\mathfrak{X}_{22})\mathfrak{X}_2 \\ \qquad + (\Omega - a_{23}\mathfrak{X}_{11} + a_{31}\mathfrak{X}_{22})\mathfrak{X}_3 = 0. \end{cases}$$

Then eliminate $\mathfrak{X}_1$, $\mathfrak{X}_2$, $\mathfrak{X}_3$ as they appear explicitly in 2) and two of the above three equations. Using the first two of the above equations, we obtain the following necessary relation

$$\begin{vmatrix} (\Omega - a_{31}\mathfrak{X}_{22} + a_{12}\mathfrak{X}_{33}), & (-a_{23}\mathfrak{X}_{22} + a_{22}\mathfrak{X}_{33}), & (-a_{33}\mathfrak{X}_{22} + a_{23}\mathfrak{X}_{33}) \\ (a_{31}\mathfrak{X}_{11} - a_{11}\mathfrak{X}_{33}), & (\Omega + a_{23}\mathfrak{X}_{11} - a_{12}\mathfrak{X}_{33}), & (a_{33}\mathfrak{X}_{11} - a_{31}\mathfrak{X}_{33}) \\ \mathfrak{X}_{11} \quad , & \mathfrak{X}_{22} \quad , & \mathfrak{X}_{33} \end{vmatrix} = 0,$$

and similar expressions for the other two possibilities. The three expressions obtained all have a common factor. The other factor is $\mathfrak{X}_{ii}$, where the i^{th} equation of 5′) is the one omitted. Hence we have, either

$$\mathfrak{X}_{11} = \mathfrak{X}_{22} = \mathfrak{X}_{33} = 0\ ,$$

or the equation obtained by placing the other factor equal zero. The latter gives

$$8')\qquad \begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & \mathfrak{X}_{11} \\ a_{21}, & a_{22}, & a_{23}, & \mathfrak{X}_{22} \\ a_{31}, & a_{32}, & a_{33}, & \mathfrak{X}_{33} \\ \mathfrak{X}_{11}, & \mathfrak{X}_{22}, & \mathfrak{X}_{33}, & 0 \end{vmatrix} - \Omega^2 = 0\,, \qquad \text{where } a_{ik} = a_{ki}\,.$$

If $\mathfrak{X}_{11} = \mathfrak{X}_{22} = \mathfrak{X}_{33} = 0$, we see from 5) that we must have $\Omega = 0$, which also follows from 8′), so that our first alternative is included in 8′) itself. Therefore 8′) is satisfied by the coordinates of all the rays of the congruence if it is to be considered as a closed continuum.

Now using the term reciprocal equation to indicate the same form of equation as that indicated by the same name in the analytic geometry, we obtain the following:

If the configuration defined by the equations 2), 3), 5) *forms a closed continuum of rays, the* $\mathfrak{X}_{ii}$ *- coordinates of the point-rays satisfy the reciprocal equation of* $S = 0$.

Using A_{ik} with the usual significance, our equation 8′) becomes,

$$A_{11}\,\mathfrak{X}_{11}^2 + A_{22}\,\mathfrak{X}_{22}^2 + A_{33}\,\mathfrak{X}_{33}^2 + 2\,A_{23}\,\mathfrak{X}_{22}\,\mathfrak{X}_{33}$$
$$+\ 2\,A_{31}\,\mathfrak{X}_{33}\,\mathfrak{X}_{11} + 2\,A_{12}\,\mathfrak{X}_{11}\,\mathfrak{X}_{22} - \Omega^2 = 0.$$

When $S\,\Delta = 0$ but *not* all its first minors zero, this becomes

$$(A_{i1}\,\mathfrak{X}_{11} + A_{i2}\,\mathfrak{X}_{22} + A_{i3}\,\mathfrak{X}_{33})^2 - A_{i1}\,\Omega = 0 \quad \text{where } A_{ik} = A_{ik}.$$

When all the first minors vanish, it is

$$\Omega^2 = 0\ .$$

Since $S = 0$ and $\Omega = 0$ are both satisfied for all the point-rays, we conclude that the field of point-rays forms a part of the configuration in this case, even as limited by 8′).

Returning to the discussion of our equations, we find that 2), 4), 8′) do not determine the same totality as 2), 4), 5) because all the rays from the system of equations given by substituting — Ω for Ω in 5) will also satisfy 2), 4), 8′).

We will now show that no other rays are given by 2), 4), 8′) than those obtained when we use both $+\Omega$ and $-\Omega$ in 2), 4), 5).

From 2) and 4) written as

$$4)\qquad \mathfrak{X}_1 S_1 + \mathfrak{X}_2 S_2 + \mathfrak{X}_3 S_3 = 0\,,$$

$$2)\qquad \mathfrak{X}_1 \mathfrak{X}_{11} + \mathfrak{X}_2 \mathfrak{X}_{22} + \mathfrak{X}_3 \mathfrak{X}_{33} = 0\,,$$

we obtain

$$\mathfrak{X}_1 : \mathfrak{X}_2 : \mathfrak{X}_3 = \begin{vmatrix} S_2 & S_3 \\ \mathfrak{X}_{22} & \mathfrak{X}_{33} \end{vmatrix} : \begin{vmatrix} S_3 & S_1 \\ \mathfrak{X}_{33} & \mathfrak{X}_{11} \end{vmatrix} : \begin{vmatrix} S_1 & S_2 \\ \mathfrak{X}_{11} & \mathfrak{X}_{22} \end{vmatrix}\,.$$

Introducing the common factor φ, we rewrite as:

$$5'')\qquad \left\{ \begin{array}{rrrrl} & -S_3\,\mathfrak{X}_{22} & +S_2\,\mathfrak{X}_{33} & +\mathfrak{X}_1\,\varphi & = 0\,, \\ S_3\,\mathfrak{X}_{11} & & -S_1\,\mathfrak{X}_{33} & +\mathfrak{X}_2\,\varphi & = 0\,, \\ -S_2\,\mathfrak{X}_{11} & +S_1\,\mathfrak{X}_{22} & & +\mathfrak{X}_3\,\varphi & = 0\,. \end{array} \right.$$

Combining 5″) and 2) as in the preceding discussion we find an equation of the form 8′) but differing from it only in that Ω^2 is replaced by φ^2. Hence in order to obtain the same system of rays in the two cases, we must have.

$$\varphi^2 = \Omega^2 \quad \text{or} \quad \varphi = \pm\,\Omega.$$

This proves our assertion.

Using $\overline{S}$ to represent the reciprocal of S, and $\underline{S}$ to indicate that the coordinates $\mathfrak{X}_{ii}$ are substituted for $\mathfrak{X}_i$ in S, we can rewrite our equations 2), 4), 8′), as

$$2)\qquad (\mathfrak{X}\,|\,\underline{\mathfrak{X}}) = 0\,,$$

$$4)\qquad S = 0\,,$$

$$8)\qquad \overline{\underline{S}} - \Omega^2 = 0\,;$$

also

$$5)\qquad S_i\,\mathfrak{X}_{jj} - S_j\,\mathfrak{X}_{ii} + \mathfrak{X}_k\,\Omega = 0\,, \qquad (i = 1,\ 2,\ 3).$$

We will now adopt the following definition:

A dual conical congruence is one the totality of whose rays is determined by the equations:

$$S = 0\,, \quad \underline{S} - \Omega^2 = 0\,, \quad (\mathfrak{X} \mid \underline{\mathfrak{X}}) = 0\,,$$

$$S_i\, \mathfrak{X}_{jj} - S_j\, \mathfrak{X}_{ii} + \mathfrak{X}_k\, \Omega = 0\,, \qquad (i = 1, 2, 3).$$

In coordinates of the second kind, we will call canonical equations those which are obtained from the canonical equations in coordinates of the first kind; namely,

$$a_{11}\, \mathfrak{X}_1{}^2 + a_{22}\, \mathfrak{X}_2{}^2 + a_{33}\, \mathfrak{X}_3{}^2 = 0\,, \tag{9}$$

$$a_{22}\, a_{33}\, \mathfrak{X}_{11}{}^2 + a_{33}\, a_{11}\, \mathfrak{X}_{22}{}^2 + a_{11}\, a_{22}\, \mathfrak{X}_{33}{}^2 - \Omega^2 = 0\,, \tag{10}$$

where

$$\Omega \equiv {}^1\!/_2\, (a_{11}\, \mathfrak{X}_1{}^2 + a_{22}\, \mathfrak{X}_2{}^2 + a_{33}\, \mathfrak{X}_2{}^2)\,. \tag{11}$$

The equations 5) become

$$\left\{\begin{array}{rrrr} & -a_{33}\, \mathfrak{X}_3\, \mathfrak{X}_{22} & + a_{22}\, \mathfrak{X}_2\, \mathfrak{X}_{33} & + \mathfrak{X}_1\, \Omega = 0\,, \\ a_{33}\, \mathfrak{X}_3\, \mathfrak{X}_{11} & & - a_{11}\, \mathfrak{X}_1\, \mathfrak{X}_{33} & + \mathfrak{X}_2\, \Omega = 0\,, \\ - a_{22}\, \mathfrak{X}_2\, \mathfrak{X}_{11} & + a_{11}\, \mathfrak{X}_1\, \mathfrak{X}_{22} & & + \mathfrak{X}_3\, \Omega = 0\,. \end{array}\right. \tag{12}$$

§ 10. Equations for the General Case of dual Parametric Representation.

We will now determine the system of equations in coordinates of the second kind, which must be satisfied when the dual coordinates are represented parametrically by functions of the second degree in two dual homogeneous variables when $S\, I \neq 0$. Take

$$X_i = \mathfrak{f}_i\, (\mathfrak{t}_1\, \mathfrak{t}_2) = f_i\, (t_1\,, t_2) +$$
$$[\varphi_i\, (t_1\,, t_2) + F_i\, (t_1\,, t_2\,; \tau_1\,, \tau_2)]\, \varepsilon\,, \qquad (i = 1, 2, 3).$$

Make $\mathfrak{t}_2$ equal a scalar t_2, (which is sufficient for our discussion), separate the functions into scalar and vector parts. Then writing $f_{i1} = {}^1\!/_2\, \dfrac{\delta f_i}{\delta t_1}\,, f_{i2} = {}^1\!/_2\, \dfrac{\delta f_i}{\delta t_2}\,,$ our representation becomes

$$X_i = f_i\, (t_1\,, t_2) + [\varphi_i\, (t_1\,, t_2) + 2 f_{i1}\, (t_1\,, t_2)\, \tau]\, \varepsilon \tag{13}$$

For the coordinates of the second kind, we have

$$\mathfrak{X}_i = f_i\,, \qquad \mathfrak{X}_{ii} = \begin{vmatrix} f_j\,; & \varphi_j \\ f_k\,, & \varphi_k \end{vmatrix} - 2 \tau\, t_2 \begin{vmatrix} f_{j1}\,, & f_{j2} \\ f_{k1}\,, & f_{k2} \end{vmatrix}\,. \tag{14}$$

Eliminating τ between the two expressions for $\mathfrak{X}_{22}$ and $\mathfrak{X}_{33}$, we obtain

$$15)\quad -\begin{vmatrix} f_{11}, f_{12} \\ f_{21}, f_{22} \end{vmatrix} \mathfrak{X}_{22} + \begin{vmatrix} f_{31}, f_{32} \\ f_{11}, f_{12} \end{vmatrix} \mathfrak{X}_{33} = -\begin{vmatrix} f_3\,, \varphi_3 \\ f_1\,, \varphi_1 \end{vmatrix} \cdot \begin{vmatrix} f_{11}, f_{12} \\ f_{21}, f_{22} \end{vmatrix} + \begin{vmatrix} f_1, \varphi_1 \\ f_2, \varphi_2 \end{vmatrix} \cdot \begin{vmatrix} f_{31}, f_{32} \\ f_{11}, f_{12} \end{vmatrix}$$

By making use of the identity

$$16)\qquad \begin{vmatrix} f_1\,, f_{11}\,, f_{12} \\ f_2\,, f_{21}\,, f_{22} \\ f_3\,, f_{31}\,, f_{33} \end{vmatrix} = 0$$

this becomes

$$-\begin{vmatrix} f_{11}, f_{12} \\ f_{21}, f_{22} \end{vmatrix} \mathfrak{X}_{22} + \begin{vmatrix} f_{31}, f_{32} \\ f_{11}, f_{12} \end{vmatrix} \mathfrak{X}_{33} = f_1 \begin{vmatrix} \varphi_1\,, f_{11}\,, f_{12} \\ \varphi_2\,, f_{21}\,, f_{22} \\ \varphi_3\,, f_{31}\,, f_{32} \end{vmatrix}$$

$$= f_1 \left\{ \varphi_1 \begin{vmatrix} f_{21}, f_{22} \\ f_{31}, f_{32} \end{vmatrix} + \varphi_2 \begin{vmatrix} f_{31}, f_{32} \\ f_{11}, f_{12} \end{vmatrix} + \varphi_3 \begin{vmatrix} f_{11}, f_{12} \\ f_{21}, f_{22} \end{vmatrix} \right\}$$

We obtain similar expressions for the other pairs $\mathfrak{X}_{33}$, $\mathfrak{X}_{11}$, and $\mathfrak{X}_{11}$, $\mathfrak{X}_{22}$.

Since we suppose here that $SI \neq 0$, we know that any homogeneous form of the second degree in t_1, t_2 can be expressed linearly and uniquely in terms of f_1, f_2, f_3. Doing this for the various functions above, then substituting in terms of $\mathfrak{X}_i$ and representing the resulting expressions by single letters as follows:

$$18)\quad S_i = \begin{vmatrix} f_{j1}, f_{j2} \\ f_{k1}, f_{k2} \end{vmatrix}, \; L_i = \varphi_i\,, \; -\Omega = L_1 S_1 + L_2 S_2 + L_3 S_3\,,$$

we obtain the following equations in coordinates of the second kind from 16) and 17).

$$19)\qquad \mathfrak{X}_1 S_1 + \mathfrak{X}_2 S_2 + \mathfrak{X}_3 S_3 = 0, \quad \text{or} \quad S = 0;$$

$$20)\quad \begin{cases} \qquad\qquad -S_3 \mathfrak{X}_{22} + S_2 \mathfrak{X}_{33} + \mathfrak{X}_1 \Omega = 0\,, \\ S_3 \mathfrak{X}_{11} \qquad\qquad -S_1 \mathfrak{X}_{33} + \mathfrak{X}_2 \Omega = 0\,, \\ -S_2 \mathfrak{X}_{11} + S_1 \mathfrak{X}_{22} \qquad\quad + \mathfrak{X}_3 \Omega = 0\,, \end{cases} \qquad \text{where } S\Delta \neq 0.$$

It can be easily verified that the expressions S_i can be written in the forms

21) $$S_i = a_{i1}\mathfrak{X}_1 + a_{i2}\mathfrak{X}_2 + a_{i3}\mathfrak{X}_3\,, \quad \text{where } a_{ik} = a_{ki}\,,$$

and therefore that

$$S_i = \frac{1}{2}\,\frac{\delta S}{\delta \mathfrak{X}_i}\,.$$

Then the equations 19), 20) are of the same type as 4), 5), so that the proper rays given by 13) or 14) belong to configurations of the type given by 2), 4), 5).

We now turn to the discussion of the point-rays given by 14). Since we suppose $SI \neq 0$, we can not find finite values for $t_1 : t_2$ for which $f_i = 0$, $(i = 1, 2, 3)$. Therefore, in order to obtain point rays, we must consider the cases for which $t_2 = 0$ and τ infinite. From 14), we see that there result perfectly definite proper rays so long as $t_2\,\tau$ is finite. If $t_2\,\tau$ becomes infinite, we consider $\mathfrak{X}_i$ as multiplied by $\sqrt{-\frac{1}{2\,t_2\,\tau}}$ and $\mathfrak{X}_{ii}$ by $-\frac{1}{2\,t_2\,\tau}$ and the limit taken. Then

22) $$\mathfrak{X}_i = 0\,, \; \mathfrak{X}_{ii} = \begin{vmatrix} f_{j1}\,, & f_{j2} \\ f_{k1}\,, & f_{k2} \end{vmatrix}\,, \qquad (i = 1,\ 2,\ 3)\,.$$

Representing the $\mathfrak{X}_i$ - coordinates of rays corresponding to the value $t = t_1 : t_2$ by $\mathfrak{X}_i{}^{(t)}$, and the point-rays corresponding to t by $0\,,\ \mathfrak{X}_{(ii)}{}^{(t)}$, we can write the following equations by applying the relations 16), 18), 21),

23) $$\left\{\begin{aligned} \mathfrak{X}_{(ii)}{}^{(t)} &= a_{i1}\mathfrak{X}^{(t)}{}_1 + a_{i2}\mathfrak{X}_2{}^{(t)} + a_{i3}\mathfrak{X}_3{}^{(t)} \\ 0 &= \mathfrak{X}_1{}^{(t)}\mathfrak{X}_{(11)}{}^{(t)} + \mathfrak{X}_2{}^{(t)}\mathfrak{X}_{(22)}{}^{(t)} + \mathfrak{X}_3{}^{(t)}\mathfrak{X}_{(33)}{}^{(t)} \end{aligned}\right.$$

Eliminating $\mathfrak{X}_i{}^{(t)}$ and dropping the index (t), we obtain the following equation which is satisfied by the point-rays obtained from 14).

24) $$\begin{vmatrix} a_{11}\,, & a_{12}\,, & a_{13}\,, & \mathfrak{X}_{(11)} \\ a_{21}\,, & a_{22}\,, & a_{23}\,, & \mathfrak{X}_{(22)} \\ a_{31}\,, & a_{32}\,, & a_{33}\,, & \mathfrak{X}_{(33)} \\ \mathfrak{X}_{(11)}\,, & \mathfrak{X}_{(22)}, & \mathfrak{X}_{(33)}, & 0 \end{vmatrix} = 0\,.$$

From equations 19), 20), 24), we see that the configuration given by 13) or 14) is of the type given by 2), 4), 5), 8); or

The configuration in the first natural continuum which is determined by the general quadratic parametric representation in dual variables is a dual conical congruence.

Extending the name synectic congruence to such congruences of the first natural continuum as are obtained from the dual representation bearing that name, we may say:

In the first natural continuum, the configuration given by the general synectic parametric representation of the second degree is a dual conical congruence, to which we give the name synectic conical congruence.

§ 11. Parallel Pencils.

Before turning to the remaining synectic quadratic configurations, we wish to make a remark upon the special type of expressions for the coordinates of all cases, and also call attention to a new grouping of the elements as they occur in the configurations and give the cross connections to the Plücker continuum for such cases as have an interpretation therein.

The expressions 14) and the discussion of the point-rays which is directly connected with 22) are in no way limited to the case $SI \neq 0$. We know immediately that there is always a point ray corresponding to a given value of $t_1 : t_2$, and its coordinates are given by 22), in which we may multiply all the expressions by any convenient finite factor. Using $\mathfrak{X}_{(ii)}$ to represent coordinates of point rays, and $\mathfrak{X}_i$, $\mathfrak{X}_{ii}$ for some proper ray of the set considered, we see that all the rays for a given value $t_1 : t_2$ are given by

25) $$\mathfrak{X}_i\,,\ \mathfrak{X}_{ii} + \lambda\,\mathfrak{X}_{(ii)}\,.$$

Such a set of rays we will call a *Parallel Pencil*[*]. The rays of such a parallel pencil all have the same direction and all intersect the same ∞^1 rays at right angles. The proof of this property will be involved in the discussion in § 17 and so will not be carried out here.

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[*] Dynamen, pp 203—4.

As coordinates of such a pencil, we will use the quantities $\mathfrak{X}_i$ and $\mathfrak{X}_{ii}$ and an additional quantity which will permit us to determine some one, although arbitrary, ray of the pencil.

Considering the quantities 25), we know from the fundamental relation 2) of coordinates of the second kind, that

$$2)\qquad \mathfrak{X}_1\,\mathfrak{X}_{11} + \mathfrak{X}_2\,\mathfrak{X}_{22} + \mathfrak{X}_3\,\mathfrak{X}_{33} = 0, \qquad \text{and also}$$

$$\mathfrak{X}_1\,\mathfrak{X}_{(11)} + \mathfrak{X}_2\,\mathfrak{X}_{(22)} + \mathfrak{X}_3\,\mathfrak{X}_{(33)} = 0,$$

must be satisfied, and therefore that

$$26)\qquad \begin{vmatrix} \mathfrak{X}_{22} & \mathfrak{X}_{33} \\ \mathfrak{X}_{(22)} & \mathfrak{X}_{(33)} \end{vmatrix} : \mathfrak{X}_1 = \begin{vmatrix} \mathfrak{X}_{33} & \mathfrak{X}_{11} \\ \mathfrak{X}_{(33)} & \mathfrak{X}_{(11)} \end{vmatrix} : \mathfrak{X}_2 = \begin{vmatrix} \mathfrak{X}_{11} & \mathfrak{X}_{22} \\ \mathfrak{X}_{(11)} & \mathfrak{X}_{(22)} \end{vmatrix} : \mathfrak{X}_3 = \Omega',$$

where Ω' is used to represent the common value of the ratios Ω' may then be considered as a coordinate of the pencil. The remaining ones, we define as follows:

$$\Xi_1 : \Xi_2 : \Xi_3 = \mathfrak{X}_1 : \mathfrak{X}_2 : \mathfrak{X}_3,$$

$$\Phi_1 : \Phi_2 : \Phi_3 = \mathfrak{X}_{(11)} : \mathfrak{X}_{(22)} : \mathfrak{X}_{(33)}.$$

These satisfy the fundamental relation

$$\Xi_1\,\Phi_1 + \Xi_2\,\Phi_2 + \Xi_3\,\Phi_3 = 0,$$

so that the seven quantities Ω', Ξ_i, Φ_i $(i = 1, 2, 3)$ are essentially but four. They have the peculiar homogeneity that if Ω' and Ξ_i or Ω' and Φ_i $(i = 1, 2, 3)$ be multiplied by the same scalar quantity, the pencil represented remains the same.

Then from the relations 14), we immediately have the theorem:

In every synectic quadratic congruence, the rays are arranged in parallel pencils, or there is but a single point-ray in the set. *

§ 12. Envelope of the Planes containing the Parallel Pencils.

For the connection with the ordinary geometry of three dimensions, in the cases where the rays belong to a Plücker continuum, $[(\mathfrak{X} \mid \mathfrak{X}) \neq 0)]$, we use the following.

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* cf. Dynamen p. 294.

Draw a plane through the origin perpendicular to the ray; its coordinates are

$$u_0 : u_1 : u_2 : u_3 = 0 : \mathfrak{X}_1 : \mathfrak{X}_2 : \mathfrak{X}_3 \,.$$

The coordinates of the point of intersection of the plane and the line are

27) $$x_0 : x_1 : x_2 : x_3 = (\mathfrak{X}|\mathfrak{X}) : \mathfrak{X}_{11} : \mathfrak{X}_{22} : \mathfrak{X}_{33} \,. \quad *$$

For all the rays of a parallel pencil, we have

$$x_0 : x_1 : x_2 : x_3 = x'_0 : x'_1 + \lambda' \mathfrak{X}_{(11)} x'_0 : x'_2 + \lambda' \mathfrak{X}_{(22)} x'_0 : x'_3 + \lambda' \mathfrak{X}_{(33)} x'_0 \,,$$

where $x'_0 : x'_1 : x'_2 : x'_3$ correspond to $\mathfrak{X}_i$, $\mathfrak{X}_{ii}$ of 25),

and where $\lambda' = \lambda \,.\, (\mathfrak{X}|\mathfrak{X})^{-1}$.

Then all the lines of the pencil lie in a plane whose coordinates are

$$u_0 : u_1 : u_2 : u_3 = \Omega' : \begin{vmatrix} \Xi_2 , & \Xi_3 \\ \Phi_2 , & \Phi_3 \end{vmatrix} : \begin{vmatrix} \Xi_3 , & \Xi_1 \\ \Phi_3 , & \Phi_1 \end{vmatrix} : \begin{vmatrix} \Xi_1 , & \Xi_2 \\ \Phi_1 , & \Phi_2 \end{vmatrix} .$$

Substituting in the coordinates of the parallel pencils, we have

$$\Xi_i = f_i \,, \quad \Phi_i = \begin{vmatrix} f_{j1} , & f_{j2} \\ f_{k1} , & f_{k2} \end{vmatrix} .$$

In the numerator of Ω', we find the same expressions as enter in the right hand members of the equations represented by 15), but with the opposite sign. Then by the same reduction and a division by f_i, we have

28) $$\Omega' = - \begin{vmatrix} \varphi_1 , & f_{11} , & f_{12} \\ \varphi_2 , & f_{21} , & f_{22} \\ \varphi_3 , & f_{31} , & f_{32} \end{vmatrix} .$$

For u_i, we have

29) $$u_0 : u_1 : u_2 : u_3 = \Omega' : \begin{vmatrix} 0 , & f_{11} , & f_{12} \\ -f_3 , & f_{21} , & f_{22} \\ f_2 , & f_{31} , & f_{32} \end{vmatrix} : \begin{vmatrix} f_3 , & f_{11} , & f_{12} \\ 0 , & f_{21} , & f_{22} \\ -f_1 , & f_{31} , & f_{32} \end{vmatrix} : \begin{vmatrix} -f_2 , & f_{11} , & f_{12} \\ f_1 , & f_{21} , & f_{22} \\ 0 , & f_{31} , & f_{32} \end{vmatrix} .$$

Writing Q_1, Q_2, Q_3 for the co-factors of the elements of the first columns, in all these determinants, which differ only in their first columns, we have

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* Dynamen p. 304 (13).

$$u_0 : u_1 : u_2 : u_3 = Q_1\,\varphi_1 + Q_2\,\varphi_2 + Q_3\,\varphi_3 :$$
$$Q_2 f_3 - Q_3 f_2 : Q_3 f_1 - Q_1 f_3 : Q_1 f_2 - Q_2 f_1.$$

Therefore:

In a synectic congruence of the second degree in a Plücker continuum, the planes of the parallel pencils envelope a ruled surface at most of the fourth class.

V. Configurations with dual Parametric Representations where $SI = 0$.

§ 13. Canonical Forms of Representation where $SI = 0$ but not all its First Minors Zero.

We have found above, § 10, that the rays determined by a dual parametric representation of the second degree with $SI \neq 0$ are rays of a configuration having a synectic equation of the second degree of the so-called general type, i. e. $S\varDelta \neq 0$. We will now take up the discussion of the nature of the configurations determined by those cases of parametric representation for which $SI = 0$, which have been excluded up to the present time. Here we will again use the form of parametric representation previously given:

1) $X_i = \mathfrak{M}_{i11}\,\mathfrak{t}_1^2 + 2\,\mathfrak{M}_{i12}\,\mathfrak{t}_1\,\mathfrak{t}_2 + \mathfrak{M}_{i22}\,\mathfrak{t}_2^2$, ($i$ = 1, 2, 3).

In order to reduce our representations to certain canonical forms more convenient for our investigation, we will apply linear synectic transformations which we represent in general by $\mathfrak{T}$, and which are defined as follows:

$\mathfrak{T}$: $\quad \mathfrak{t}_1 = \mathfrak{a}\,\mathfrak{t}_1' + \mathfrak{b}\,\mathfrak{t}_2', \quad \mathfrak{t}_2 = \mathfrak{c}\,\mathfrak{t}_1', + \mathfrak{d}\,\mathfrak{t}_2'$;

and we call $\mathfrak{D}$ its determinant, where

$$\mathfrak{D} \equiv \mathfrak{a}\mathfrak{d} - \mathfrak{b}\mathfrak{c}.$$

After applying the transformation, we will omit the primes, and then, as in the preceding section, § 10, make $\mathfrak{t}_2$ equal the scalar t_2.

For our present purpose, it is only necessary for us to follow the transformation of the scalar part more in detail. This we write

2) $$\mathfrak{X}_{oi} = M_{i11}\, t_1^2 + 2\, M_{i12}\, t_1\, t_2 + M_{i22}\, t_2^2\,, \quad (i = 1, 2, 3)\,.$$

Further from the nature of these dual quantities, we find that the scalar part of the representation resulting from the application of $\mathfrak{T}$ to 1) is the same as that obtained by applying T, the scalar part of $\mathfrak{T}$, to 2), where

$$T: \qquad t_1 = a\, t_1' + b\, t_2'\,, \; t_2 = c\, t_1' + d\, t_2'.$$

We can further separate such representations into two classes for which the corresponding configurations have essentially different characters: namely, A) where $SI = 0$ but *not* all first minors zero, and B), where $SI = 0$ and all first minors zero. We will now discuss these two classes separately, in each case first reducing to canonical form.

In class A), we make a still further division into two classes: α) where the three functions f_i do not have a common factor, and β) where they have one common factor. Then by properly choosing $\mathfrak{T}$, with $S\mathfrak{D} \neq 0$, we can reduce to the following types, neither of which can be transformed into the other.

3) $$\begin{aligned} \alpha)\quad \mathfrak{X}_{oi} &= M_{i11}\, t_1^2 + M_{i22}\, t^2_2\,, \\ \beta)\quad \mathfrak{X}_{oi} &= (2\, M_{i12}\, t_1 + M_{i22}\, t_2)\, t_2\,, \end{aligned} \qquad (i = 1, 2, 3).$$

where the quantities M_{ijk} are not all zero.

In both of these cases, we know that the coordinates $\mathfrak{X}_{oi}$ satisfy a linear relation, as

4) $$N_1\, \mathfrak{X}_{01} + N_2\, \mathfrak{X}_{02} + N_3\, \mathfrak{X}_{03} = 0\,,$$

since $SI = 0$. Hence

All the rays given by a parametric representation of classes A) and B) form right angles with a definite direction and therefore belong to a planar complex.

Then they all satisfy an equation of the form

$$(\mathfrak{N}_{i1} X_i) \cdot (\mathfrak{N}_{i2} X_i) = 0, \text{ in which } S\mathfrak{N}_{ij} = N_i\,, \quad \left(\begin{smallmatrix} i = 1,\, 2,\, 3 \\ j = 1,\, 2 \end{smallmatrix}\right);$$

where we define

$$(\mathfrak{N}\, X) \equiv (\mathfrak{N}_i\, X_i) \equiv (\mathfrak{N}_1\, X_1 + \mathfrak{N}_2\, X_2 + \mathfrak{N}_3\, X_3)\,.$$

In case α), *we obtain two sets of rays which belong to any one of the admissible directions;* since there are two values of $t_1 : t_2$ for every admissible set of values $\mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03}$. In special cases the two sets will coincide for all values of $t_1 : t_2$, but in general they do not.

In case β), *every ray is obtained but once.* For admissible values of $\mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03}$ for which not all the terms are zero, we have two values of $t_1 : t_2$, but in one of them is $t_2 = 0$. This however gives $\mathfrak{X}_{oi} = 0$, $(i = 1, 2, 3)$, and for values depending on $t_1 : t_2$ only, this is the only one giving this set of values; because otherwise our quadratic forms are such that $SI = 0$ and all its first minors zero, and this is expressly excluded in this discussion.

In general, $\mathfrak{X}_{jk}$ $(j, k = 1, 2, 3,\ j \neq k)$, contain both t_1^2 and $t_1 t_2$ in the part independent of τ, and a transformation $\mathfrak{T}$ can not be found such that $t_1 t_2$ in α) or t_1^2 in β) shall disappear from all the expressions $\mathfrak{X}_{jk}$ and still reduce the scalar parts to the preceding types respectively. The cases for which it is possible to perform the above mentioned elimination give two special types of the present class for both of which $I = 0$.

The former gives the canonical form

$$5)\qquad \alpha_2)\quad X_i = \mathfrak{M}_{i11}\,\mathfrak{t}_1^2 + \mathfrak{M}_{i22}\,\mathfrak{t}_2^2\,, \qquad (i = 1, 2, 3),$$

which is still quadratic. For the other we have

$$5)\qquad \beta_2{}')\quad X_i = (2\,\mathfrak{M}_{i12}\,\mathfrak{t}_1 + \mathfrak{M}_{i22}\,\mathfrak{t}_2)\,\mathfrak{t}_2\,, \qquad (i = 1, 2, 3)\,,$$

which is essentially linear.

In case α), all the synectic parametric representations for which $I = 0$ are reducible to the form $5\,\alpha_2$). This we prove as follows:

Taking the representation 1) and applying the transformation $\mathfrak{T}$ for which $S\,\mathfrak{D} \neq 0$, and imposing the condition that $M_{i12}{}' = 0$, $(i = 1, 2, 3)$, we find that

$$\text{either}\qquad a = d = 0\,, \qquad \text{or} \qquad c = b = 0.$$

Then, writing εM_{ijk} for $V\mathfrak{M}_{ijk}$, we must have

$$M'_{i12} = M_{i11}\,a\,b + M_{i22}\,c\,\delta + M_{i12}\,b\,c\,, \qquad \text{or}$$

$$M'_{i12} = M_{i11}\,a\,\beta\, M_{i22}\,\gamma\, d + M_{i12}\,a\,d\,, \qquad (i = 1, 2, 3),$$

respectively. In either case, the necessary and sufficient condition for the vanishing of M_{i12} $(i = 1, 2, 3)$ is the vanishing of the same determinant, namely that of the quantities M_{i11}, M_{i22}, M_{i12}. This, however, is the necessary and sufficient condition for the vanishing of I.

In case β), when $I = 0$, we know from $VI = 0$, that

$$M_{i11} = 4 c_1 M_{i12} + 2 c_2 M_{i22}, \qquad (i = 1, 2, 3),$$

or, on account of the dual homogeneity of the representation, we may say that the form is such that

$$M_{i11} = c_2 M_{i22}, \qquad (i = 1, 2, 3),$$

and the general case under this condition is represented by,

$$5\beta_2) \qquad X_i = (2\,\mathfrak{M}_{i12}\,\mathfrak{t}_1 + \mathfrak{M}_{i22}\mathfrak{t}_2)\,\mathfrak{t}_2 + \varepsilon c_2 M_{i22} t_1^2, \quad (i = 1, 2, 3).$$

This can not be reduced to $5\beta_2'$) by transformations $\mathfrak{T}$ with $S\mathfrak{D} \neq 0$. For the representations of this class, in order that $M_{i11} = 0$ $(i = 1, 2, 3)$, we must have $c = 0$, and then

$$M'_{i11} = 2 M_{i12}\, a\gamma + a^2 c_2 M_{i22}, \qquad (i = 1, 2, 3);$$

and since $a \neq 0$, this can not be reduced to $5\beta_2'$) unless $c_2 = 0$, i. e. unless it was originally in that form, since otherwise all the first minors of SI must vanish.

We will now, however, exclude all cases β_2) from our further discussion for the following reasons. In $5\beta_2'$) the expressions all have a common factor $\mathfrak{t}_2$, and then become linear after division by it, and the original representation was not, properly speaking, quadratic. The reason for the exclusion of the remaining cases, is that they also give rays which intersect a single ray at right angles, as will appear more clearly after the discussion of the point rays and of the configurations which have a correspondence in the Euclidean space. (§ 14). Every ray is counted but once, as we saw above; and all belong to a congruence satisfying a linear dual equation, as will be seen from § 17.

§ 14. Coordinates of the Second Kind. Geometrical Description of the Configurations for Case *A*).

For the further discussion of these cases, we will introduce coordinates of the second kind. Then as before, we have

$$6)\qquad \mathfrak{X}_i = f_j\,,\quad \mathfrak{X}_{ii} = \begin{vmatrix} f_j\,, & \varphi_j \\ f_k\,, & \varphi_k \end{vmatrix} - 2\tau\, t_2 \begin{vmatrix} f_{j1}, & f_{j2} \\ f_{k1}, & f_{k2} \end{vmatrix}, \ (i = 1, 2, 3).$$

For point rays, we must have $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}_3 = 0$ or values reducible to this form by using the homogeneity of the system.

For case α), we can not obtain $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}_3 = 0$ for any set of values depending upon $t_1 : t_2$ alone where both t_1 and t_2 are not zero, or one in which τ is finite. Now dividing the expressions for $\mathfrak{X}_i$ by $(-2\tau\, t_1\, t_2^2)^{1/2}$ and those for $\mathfrak{X}_{ii}$ by $-2\tau t_1 t_2^2$ and taking the limit for $\tau t_1 t_2^2$ infinite, we obtain

$$\mathfrak{X}_i = 0,\ \mathfrak{X}_{ii} = \begin{vmatrix} M_{j11}\,, & M_{j22} \\ M_{k11}\,, & M_{k22} \end{vmatrix}, \qquad (i = 1, 2, 3),$$

which gives but a single point ray.

For case β), we found above that $t_2 = 0$ gave $\mathfrak{X}_1 = {}_2\mathfrak{X} = \mathfrak{X}_3 = 0$, but we find that in this case $\mathfrak{X}_{11} = \mathfrak{X}_{22} = \mathfrak{X}_{33}$ are also zero. Now dividing $\mathfrak{X}_i$ by $t_2^{1/2}$ and $\mathfrak{X}_{ii}$ by t_2, we obtain

$$7)\qquad \mathfrak{X}_i = (2M_{i12}\, t_1 + M_{i22}\, t_2)\, t_2^{1/2},$$

$$\mathfrak{X}_{ii} = \begin{vmatrix} 2\,M_{j12}\, t_1 + M_{j22}\, t_2\,, & \varphi_j \\ 2\,M_{k12}\, t_1 + M_{k22}\, t_2\,, & \varphi_k \end{vmatrix} - 2\tau t_2^2 \begin{vmatrix} M_{j12}\,, & M_{j22} \\ M_{k12}\,, & M_{k22} \end{vmatrix}, (i = 1, 2, 3).$$

Using εM_{ijk} to designate $V\mathfrak{M}_{ijk}$ as before, passing to the limit for $t_2 = 0$, and performing a division by 2 in $\mathfrak{X}_{ii}$, we obtain

$$8)\qquad \mathfrak{X}_i = 0\,,\ \mathfrak{X}_{ii} = t_1^3 \begin{vmatrix} M_{j12}\,, & M_{j11} \\ M_{k12}\,, & M_{k11} \end{vmatrix} - \begin{vmatrix} M_{j12}\,, & M_{j22} \\ M_{k12}\,, & M_{k22} \end{vmatrix} . \lim_{t_2 = 0}. (\tau t_2^2)\,, \quad (i = 1, 2, 3).$$

In this case, we obtain an infinite number of point rays unless $M_{i11} = 4c_1\, M_{i12} + 2c_2\, M_{i22}$ $(i = 1, 2, 3)$, in which case they coincide with the one obtained for values independent of $t_1 : t_2$, namely where τt_2^3 is infinite. Since, dividing by $(-2\,\tau t_2^3)^{1/2}$ and $-2\tau t_2^3$ (instead of as in 7), and taking the limit, we have

$$\mathfrak{X}_i = 0, \qquad \mathfrak{X}_{ii} = \begin{vmatrix} M_{j12}\,, & M_{j22} \\ M_{k22}\,, & M_{k22} \end{vmatrix}, \qquad (i = 1, 2, 3).$$

This point ray belongs to every set of parallel rays of the congruence, whatever the direction.

Recalling the reason for the introduction of coordinates of the second kind, we see by referring to 27) § 12, that the point-rays and the points on the plane at infinity have a one-to-one correspondence with each other and, in fact, can be considered as essentially the same.* We may then speak of point-rays as lying on loci in the plane at infinity just as we speak of the points in that plane. Using this idea and also that of parallel pencil as introduced in § 11, we may summarize as follows:

In class A), the rays having the same direction lie in two parallel pencils which are different in case α), *and are coincident in case* α_2)*; and but one for every direction in case* β). *In both cases, all the parallel pencils have the same point-ray in common: in* α), *there are no other point-rays*; *in* β), *there are* ∞^1 *other point-rays, all on the same straight line in the plane at infinity.*

Using $l = 1$ or 2 according as we have case α) or β), the coordinates of the parallel pencils are

$$\Xi_i = f_i\,, \quad \Phi_i = \begin{vmatrix} M_{j1l}\,, & M_{j2l} \\ M_{k1l}\,, & M_{k2l} \end{vmatrix}, \ (i = 1,\ 2,\ 3)\,;$$

and for Ω', substituting in the expression 28) § 12, we obtain

$$\Omega' = -\begin{vmatrix} \varphi_1, & M_{11l}, & M_{122} \\ \varphi_2, & M_{21l}, & M_{222} \\ \varphi_3, & M_{31l}, & M_{322} \end{vmatrix}.$$

In order to obtain an interpretation in ordinary Euclidean space for such cases as have a representation therein, we substitute in 29) § 12, and obtain for the coordinates of the planes containing the parallel pencils:

$$u_0 : u_1 : u_2 : u_3 =$$

$$\Omega' : \begin{vmatrix} 0, & M_{11l}, & M_{122} \\ -f_3, & M_{21l}, & M_{222} \\ f_2, & M_{31l}, & M_{322} \end{vmatrix} : \begin{vmatrix} f_3, & M_{11l}, & M_{122} \\ 0, & M_{21l}, & M_{222} \\ -f_1, & M_{31l}, & M_{322} \end{vmatrix} : \begin{vmatrix} -f_2, & M_{11l}, & M_{122} \\ f_1, & M_{21l}, & M_{222} \\ 0, & M_{31l}, & M_{322} \end{vmatrix};$$

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* Dynamen p. 261.

or, designating the co-factors of the elements of the first column of all these determinants by k_1, k_2, k_3 respectively.

$$u_0 : u_1 : u_2 : u_3 =$$
$$k_1 \varphi_1 + k_2 \varphi_2 + k_3 \varphi_3 : k_2 f_3 - k_3 f_2 : k_3 f_1 - k_1 f_3 : k_2 f_1 - k_1 f_2 .$$

By substitution, we can verify that

$$k_1 u_1 + k_2 u_2 + k_3 u_3 = 0 ,$$

and therefore all the planes are parallel to the lines whose direction-cosines are proportional to $k_1 : k_2 : k_3$. The expressions for u_i are all quadratic functions of $t_1 : t_2$; which all have a common factor t_2, in case β. In case α_2), the term $t_1 t_2$ does not appear in the expression for Ω', and the coordinates u_i become linear functions of $t_1^2 : t_2^2$; in case β_2), Ξ_i and Ω' all have a common factor t_2, and on account of the homogeneity belonging to the system, we can divide by it. Then Ξ_i, Ω' become linear in $t_1 : t_2$, and as Φ_i are all constant, we consider the representation as not properly quadratic and so to be excluded, as has already been done in § 13.

We will now collect our results after giving the following definition.

A cylindrical congruence is the totality of all those tangents to a cylinder which are perpendicular to the generators passing through the points of contact.

The rays of the quadratic synectic congruences of Class A) form a cylindrical congruence, in which the cylinder is of the second class. In case α), the cylinder is elliptic or hyperbolic; in case α_2), it is a straight line counted twice: and in case β), it is parabolic.

§ 15. Canonical Forms when all the First Minors of SI are zero. Geometrical Description of the Configurations.

In class B), from the condition that all the first minors of SI are zero, the functions f_i must all contain the same factors. Again we divide into two classes; α) where the two factors are different, β) where they coincide. The two cases can then be reduced to the canonical forms:

$$9)\qquad \begin{cases} \alpha) & \mathfrak{X}_{oi} = M_{i12}\, t_1 t_2\,, \\ \beta) & \mathfrak{X}_{oi} = M_{i22}\, t_2^2\,, \end{cases} \qquad (i = 1,\ 2,\ 3)\,.$$

Writing the representations some-what more at length, we have for the two cases, respectively:

$$10)\qquad \begin{cases} \alpha) & X_i = M_{i12}\, t_1 t_2 + (\varphi_i + M_{i12}\, t_2 \tau)\, \varepsilon\,, \\ \beta) & X_i = M_{i22}\, t_2^2 + \varphi_i \,.\, \varepsilon\,, \end{cases} \qquad (i = 1, 2, 3)\,.$$

For the case α), however, on account of the dual homogeneity, we can drop the term $M_{i12}\, t_2\, \tau \varepsilon$ in X_i, and can then rewrite in a single form for the two cases

$$11)\qquad X_i = M_i\, t_l\, t_2 + \varphi_i \,.\, \varepsilon\,, \qquad (i = 1,\ 2,\ 3)\,,$$

where l is 1 or 2 according as we have case α) or β), respectively. From this system of equations, we can say:

In class B), there are only ∞^1 *proper rays and they are all parallel.*

It follows from 11), that all the rays given by this representation satisfy the synectic equations represented by

$$(\mathfrak{N}_1\, X)\,.\,(\mathfrak{N}_2\, X) + \varepsilon\, (\mathfrak{N}_3\, X)\,.\,(\mathfrak{N}_4\, X) = 0\,,$$

where the quantites $\mathfrak{N}$ satisfy the conditions

$$N_{i1} M_1 + N_{i2} M_2 + N_{i3} M_3 = 0\,, \qquad (i = 1,\ 2,\ 3)\,,$$

and $\mathfrak{N}_{41}$, $\mathfrak{N}_{42}$, $\mathfrak{N}_{43}$ are entirely arbitrary.

The coordinates of the second kind are given by the equations

$$12)\qquad \mathfrak{X}_i = M_i\, t_l\, t_2\,,\ \mathfrak{X}_{ii} = \begin{vmatrix} M_j\,, & \varphi_j \\ M_k\,, & \varphi_k \end{vmatrix} t_l\, t_2\,, \qquad (i = 1,\ 2,\ 3)\,,$$

or, after dividing $\mathfrak{X}_i$ by $(t_l\, t_2)^{1/2}$ and $\mathfrak{X}_{ii}$ by $t_l\, t_2$,

$$\mathfrak{X}_i = M_i \sqrt{t_l\, t_2}\,,\quad \mathfrak{X}_{ii} = \begin{vmatrix} M_j\,, & \varphi_j \\ M_k\,, & \varphi_k \end{vmatrix}\,, \quad (i = 1,\ 2,\ 3)\,.$$

Taking the limit for $t_l\, t_2$ equal zero, we find the coordinates of the point-rays. In case α), we have two point rays, one for each of the values $t_1 = 0$, and $t_2 = 0$, except when $M_{i22} = c\, M_i$ and $M_{i11} = c\, M_i$ $(i = 1,\ 2,\ 3)$, respectively. In either of these cases, the M corresponding can be made to vanish by multiplication by a properly chosen dual factor, and the parametric representation 11) reduces by dividing by the corresponding t. It is then no longer a quadratic representation and is excluded. In case β), we obtain one point-ray, unless $M_{i11} = cM_i$ ($i = 1, 2, 3$). Here, as in the preceding case, the representation reduces under these circumstances, and is excluded.

Summarizing we have:

For the properly quadratic parametric representations of class B), we have two point rays in case α), *which coincide if* $M_{i11} = cM_{i22}$ ($i = 1, 2, 3$)*; and but one point ray in case* β).

We will now pass to the Euclidean geometry, and determine the nature of the totalities of lines given by such representations. We introduce the following relations already given in § 12.

$$x_0 : x_1 : x_2 : x_3 = (\mathfrak{X}/\mathfrak{X}) : \mathfrak{X}_{11} ; \mathfrak{X}_{22} : \mathfrak{X}_{33} .$$

From 12), by cancelling the factor $t_l\, t_2$, we have

$$x_0 : x_1 : x_2 : x_3 = t_l\, t_2\, (M/M) : \begin{vmatrix} M_2\,, & \varphi_2 \\ M_3\,, & \varphi_3 \end{vmatrix} : \begin{vmatrix} M_3\,, & \varphi_3 \\ M_1\,, & \varphi_1 \end{vmatrix} : \begin{vmatrix} M_1\,, & \varphi_1 \\ M_2\,, & \varphi_2 \end{vmatrix} .$$

We also find that

$$M_1\, x_1 + M_2\, x_2 + M_3\, x_3 = 0 ,$$

and we conclude that the points x_i all lie on a curve of the second order, or on a straight line counted twice. Hence

The proper rays given by a properly quadratic representation of the class B) are the generators of a non-degenerate cylinder of the second order, or they form the rays of a parallel plane pencil counted twice.

VI. General Descriptive Properties of Dual Conical Congruences.

§ 16. Discussion of the quadratic Equation in a single dual Variable.

Before taking up the discussion of properties of the dual conical congruences, it will be necessary to introduce some preliminary considerations which have to do, 1) with the solutions of equations in a single dual variable, in particular, with the quadratic equation; 2) with the configurations of the ray geometry which are represented by linear homogeneous equations in dual variables or their corresponding equations in coordinates of the second kind.

Let $\mathfrak{f}_n(\mathfrak{x})$ represent*

$$\begin{aligned}\mathfrak{f}_n(\mathfrak{x}) &\equiv \mathfrak{a}_0\,\mathfrak{x}^n + \mathfrak{a}_1\,\mathfrak{x}^{n-1} + \mathfrak{a}_2\,x^{n-2} + \cdots\cdots + \mathfrak{a}_n\\ &= f(x) + [f'(x) \,.\, \xi + \varphi(x)]\,\varepsilon,\end{aligned}$$

where

$$f(x) \equiv a_0\,x^n + a_1\,x^{n-1} + a_2\,x^{n-2} + \cdots\cdots + a_n$$

$$\varphi(x) \equiv \alpha_0\,x^n + \alpha_1\,x^{n-1} + \alpha_2\,x^{n-2} + \cdots\cdots + \alpha^n\,.$$

If $\mathfrak{f}(\mathfrak{x}) = 0$, then both $f(x) = 0$ and $f'(x) \,.\, \xi + \varphi(x) = 0$; and we may have the following cases:

1) $f'(x) \neq 0$. Then ξ has entirely definite values.

2) $f'(x) = 0$. If $\varphi(x) \neq 0$, ξ can not have finite values; but if $\varphi(x) = 0$ also, ξ is entirely indeterminate.

Turning now to the quadratic equation, we will find the condition in order that the ratio of the two roots shall be scalar.

We write the resulting equations as

1) $$a_0\,x^2 + a_1\,x + a_2 = 0\,.$$

2) $$(2\,a_0\,x + a_1)\,\xi + (\alpha_0\,x^2 + \alpha_1\,x + \alpha_2) = 0.$$

Calling x_1, x_2, ξ_1, ξ_2, the values of x and ξ which satisfy these equations, the ratio of the two values $\mathfrak{x}_1 : \mathfrak{x}_2$ is

$$\mathfrak{x}_1 : \mathfrak{x}_2 = x_1 + \xi_1\,\varepsilon : x_2 + \xi_2\,\varepsilon\,.$$

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* Dynamen pp. 197—198.

For this to be scalar, the necessary and sufficient condition is

$$x_2 \xi_1 - x_1 \xi_2 = 0, \quad \text{or} \quad x_1 : x_2 = \xi_1 : \xi_2 .$$

Substituting in this equation from 1) and 2) and rewriting in terms of the coefficients, this becomes

$$3) \qquad \frac{a_1 a_2 a_0 + a_0 a_1 a_2 - 2 a_0 a_2 a_1}{a_0^2 \sqrt{a_1^2 - 4 a_0 a_2}} = 0.$$

As we will consider that the coefficients are all finite, we have a necessary condition, as

$$4) \qquad a_1 a_2 a_0 + a_0 a_1 a_2 - 2 a_0 a_2 a_1 = 0;$$

and as long as $a_0 (a_1^2 - 4 a_0 a_2) \neq 0$, it is also sufficient. For $a_0 = 0$, and 4) also satisfied, the ratio is either a finite scalar or infinite. However, when 4) is satisfied and also $a_1^2 - 4 a_0 a_2 = 0$, (except when $a_1 = a_2 = 0$ or $a_0 = a_2 = 0$), ξ has no definite value, as these are the conditions under which both terms in 2) vanish; similarly when $a_1 = a_2 = 0$ or $a_0 = a_1 = 0$, and $a_0 = a_1 = a_2 = 0$, ξ is indefinite. Further cases will not be discussed, as we need consider only cases where $a_0 \neq 0$.

§ 17. Normal Nets.

We will now consider the linear dual equation*

$$5) \qquad \mathfrak{L} \equiv (UX) \equiv U_1 X_1 + U_2 X_2 + U_3 X_3 = 0 .$$

This expresses the condition that the ray X_i intersects the ray U_i at right angles. If U_i $(i = 1, 2, 3)$ are constants, this equation defines the totality of rays X_i which cut U_i at right angles, thus giving a congruence to which has been given the name *Normal Net.* The ray U_i is called the *Principal Axis* or *Directrix* of the normal net. The quantities U_i may be considered as the coordinates of the normal net, and then we obtain the same kind of reciprocal relation, in this geometry, between ray and normal net in combined position as holds in plane geometry between point and line in similar relation to each other, and also a corresponding relation between the dual collineations.

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* Dynamen p. 202.

Taking the coordinates of two rays as X_1 and X_2, and forming

6) $$\mathfrak{l}_1 X_1 + \mathfrak{l}_2 X_2 ,$$

where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are dual quantities, we find that if X_1 and X_2 satisfy 5), then all the rays given by 6) also belong to 5). Also, if X_1 and X_2 are not parallel and satisfy the equation 5), then all the proper rays of the normal net are given by 6). Further all the proper rays of a normal net 5) which have the same direction can be obtained from any two of them by the formula 6), where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are *scalar*. Also that the equation of a normal net determined by two non-parallel rays X_1 and X_2 is

$$(UX) \equiv (X_1 X_2 X) = \begin{vmatrix} X_{11}, & X_{12}, & X_{13} \\ X_{21}, & X_{22}, & X_{23} \\ X_1, & X_2, & X_3 \end{vmatrix} = 0 .$$

Writing $U = u + v\,\varepsilon$, and forming the equations of the normal net 5) in coordinates of the second kind, we have

7) $$L \equiv u_1 \mathfrak{X}_1 + u_2 \mathfrak{X}_2 + u_3 \mathfrak{X}_3 = 0 ,$$

8) $$\left| \begin{array}{rrrl} & -u_3 \mathfrak{X}_{22} & + u_2 \mathfrak{X}_{33} & + \mathfrak{X}_1 \omega = 0 , \\ u_3 \mathfrak{X}_{11} & & - u_1 \mathfrak{X}_{33} & + \mathfrak{X}_2 \omega = 0 , \\ -u_2 \mathfrak{X}_{11} & + u_1 \mathfrak{X}_{22} & & + \mathfrak{X}_3 \omega = 0 , \end{array} \right.$$

where $$-\omega \equiv v_1 \mathfrak{X}_1 + v_2 \mathfrak{X}_2 + v_3 \mathfrak{X}_3 .$$

We find a single point-ray in this configuration, namely,

$$\mathfrak{X}_i = 0, \qquad \mathfrak{X}_{ii} = u_i , \qquad (i = 1, 2, 3) ,$$

which belongs to the family of normal nets for which $S\,U_i = u_i$, $(i = 1, 2, 3)$, i. e. which have parallel directrices. We may also say that the point-rays are determined by the directions in which they lie, and from the above we see that the coordinates $\mathfrak{X}_{(ii)}$ $(i = 1, 2, 3)$ give the direction.

Now forming the coordinates of the whole family of parallel rays obtained from the two given by $\mathfrak{X}_i$, $\mathfrak{X}_{ii}^{(1)}$ and $\mathfrak{X}_i$, $\mathfrak{X}_{ii}^{(2)}$ we find them to be

$$\mathfrak{X}_i = \mathfrak{X}_i , \quad \mathfrak{X}_{ii} = \frac{l_1 \mathfrak{X}_{ii}^{(1)} + l_2 \mathfrak{X}_{ii}^{(2)}}{l_1 + l_2} .$$

This may be written

$$\mathfrak{X}_i = \mathfrak{X}_i\,, \quad \mathfrak{X}_{ii} = \mathfrak{X}_{ii}^{(1)} + \frac{l_2}{l_1 + l_2}\,(\mathfrak{X}_{ii}^{(2)} - \mathfrak{X}_{ii}^{(1)})\,, \quad (i = 1,\ 2,\ 3)\,,$$

which is of the form

$$9) \qquad \mathfrak{X}_i = \mathfrak{X}_i\,, \quad \mathfrak{X}_{ii} = \mathfrak{X}_{ii}^{(1)} + \lambda\,\mathfrak{X}_{(ii)}$$

already used in § 11 in connection with parallel pencils. We have

$$\lambda = \frac{l_2}{l_1 + l_2} \quad \text{and} \quad \mathfrak{X}_{(ii)} = \mathfrak{X}_{ii}^{(1)} - \mathfrak{X}_{ii}^{(2)}\,.$$

From equations 8), $\mathfrak{X}_{(ii)}$ are the coordinates of the point-ray of the net 5) and therefore $u_i = \mathfrak{X}_{(ii)}$; from the same equations, we see that $\omega = \Omega'$ of 26) § 11.

We see that the rays 9) belong to the normal net whose directrix is

$$u_i = u_i\,, \quad u_{ii} = \begin{vmatrix} u_j\,, & v_j \\ u_k\,, & v_k \end{vmatrix}\,, \qquad (i = 1,\ 2,\ 3)\,.$$

Indeed, they belong to all those normal nets whose directrices are

$$u_i\,,\ u_{ii} + \mu\mathfrak{X}_i \ \text{or}\ U_{oi} = u_i\,,\ U_{jk} = v_i + \mu' \begin{vmatrix} u_j\,, & \mathfrak{X}_j \\ u_k\,, & \mathfrak{X}_k \end{vmatrix}, \ (i = 1, 2, 3)\,.$$

Hence we have the result, that

All the rays of a parallel pencil belong to a family of normal nets whose directrices also form a parallel pencil.

We will give the name *proper normal net* or simply *normal net* to such as have a proper ray for directrix.

§ 18. The Rays common to a dual Conical Congruence and a Normal Net. Tangent Normal Net.

We will now turn to the discussion of the conical congruences, and will first prove the following property.

If a dual conical congruence contains two parallel proper rays, it contains all the rays of the parallel pencil determined by them.

We will use the dual equation. Then

$$\tfrac{1}{2}\Sigma \equiv S_1\,\mathfrak{X}_{23} + S_2\,\mathfrak{X}_{31} + S_3\,\mathfrak{X}_{12} = 0.$$

Let the two parallel rays be given by $\mathfrak{X}_{oi} + \varepsilon\,\mathfrak{X}_{jk}$ and $\mathfrak{X}_{oi} + \varepsilon\,\mathfrak{X}'_{jk}$, $(i = 1, 2, 3)$. Then all the rays of the parallel pencil are given by

$$(l + l')\,\mathfrak{X}_{oi} + \varepsilon\,(l\,\mathfrak{X}_{jk} + l'\,\mathfrak{X}'_{jk}), \qquad (i = 1, 2, 3).$$

Substituting in Σ, we obtain

$$(l + l')\,(l\,\Sigma + l'\,\Sigma') = 0;$$

since $\Sigma = 0$ and $\Sigma' = 0$ by hypothesis.

From this it follows as a corollary that

If a dual conical congruence contains three parallel rays not in the same parallel pencil, it contains all the rays of the bundle to which they belong.

Also

The parallel rays of a dual conical congruence can be arranged in parallel pencils; as we have proved before for the dual parametric representations only. Since to every admissible set of values $\mathfrak{X}_i \neq 0$ $(i = 1, 2, 3)$, we have an infinite number of rays which belong to the congruence.

We will now prove the theorem:

Every normal net has at least two rays in common with a dual conical congruence.

Take two rays of the normal net which are not parallel, as X_1 and X_2. Substituting $\mathfrak{l}_1 X_1 + \mathfrak{l}_2 X_2$ for X in the equation $\mathfrak{S} = 0$, we obtain

10) $$\mathfrak{l}_1^2\,\mathfrak{S}^{(1)} + 2\,\mathfrak{l}_1\mathfrak{l}_2\,\mathfrak{P}_{12} + \mathfrak{l}_2^2\,\mathfrak{S}^{(2)} = 0$$

for the determination of $\mathfrak{l}_2 : \mathfrak{l}_1$; where $\mathfrak{S}^{(1)}$ and $\mathfrak{S}^{(2)}$ are the results of substituting X_1 and X_2 respectively for X in $\mathfrak{S}$; and, after writing

$$\mathfrak{S}_i = \frac{1}{2}\,\frac{\delta\mathfrak{S}}{\delta\mathfrak{X}_i},$$

11) $$\mathfrak{P}_{12} = X_{21}\,\mathfrak{S}_1^{(1)} + X_{22}\,\mathfrak{S}_2^{(1)} + X_{23}\,\mathfrak{S}_3^{(1)} = \frac{1}{2}\left(X_2\,\frac{\delta\mathfrak{S}^{(1)}}{\delta\mathfrak{X}_1}\right) = \frac{1}{2}\left(X_1\,\frac{\delta\mathfrak{S}^{(2)}}{\delta\mathfrak{X}_2}\right).$$

Representing the discriminant of 10) by D, we have

$$\mathsf{D} = \mathfrak{P}_{12}{}^2 - \mathfrak{S}^{(1)}\,\mathfrak{S}^{(2)} \text{ and } S\mathsf{D} = P_{12}{}^2 - S^{(1)}\,S^{(2)} .$$

For our discussion, it is convenient to choose $\mathfrak{X}_{oi}{}^{(1)}$ and $\mathfrak{X}_{oi}{}^{(2)}$ in a special relation to each other, namely, so that $P_{12} = 0$, but $S^{(2)} \neq 0$. Then for the discriminant of the scalar part of 10), we have

$$S\mathsf{D} = S^{(1)}\,S^{(2)} .$$

From the condition that $S^{(2} \neq 0$, we know that l_1 is arbitrary and can be taken different from zero. Placing $\mathfrak{l}_2 : \mathfrak{l}_1 = \mathfrak{l}$ in the equation, we know from our previous discussion of the quadratic dual equation (§ 16), that so long as $S^{(1)} \neq 0$, we have two and only two different values of $\mathfrak{l}$, and therefore there are two different non-parallel rays common to the two configurations.

If $S^{(1)} = 0$, we have two cases according as $\Sigma^{(1)}$ is or is not zero.

If $\Sigma^{(1)} = 0$, X_1 is a ray of the congruence and we obtain an infinite number of values for $\mathfrak{l}$, all of which have the same scalar part, $l = 0$. Then the normal net has an infinite number of parallel rays in common with the congruence. Their coordinates are

$$l_1\, X_1 + \varepsilon\, \lambda_2\, X_2 , \tag{12}$$

and they all belong to the normal net

$$\mathfrak{P} \equiv X_1\, \mathfrak{S}_1{}^{(1)} + X_2\, \mathfrak{S}_2{}^{(1)} + X_3\, \mathfrak{S}_3{}^{(1)} = 0 \tag{13}$$

This normal net, we will call a *Tangent Normal Net* of the dual conical congruence at the ray X_1. In fact these rays 12) belong to every net of the family whose equations are

$$X_1 \left(\mathfrak{S}_1{}^{(1)} + \varepsilon\mu \begin{vmatrix} S_2{}^{(1)}, \mathfrak{X}_{02}{}^{(1)} \\ S_3{}^{(1)}, \mathfrak{X}_{03}{}^{(1)} \end{vmatrix} \right) + X_2 \left(\mathfrak{S}_2{}^{(1)} + \varepsilon\mu \begin{vmatrix} S_3{}^{(1)}, \mathfrak{X}_{03}{}^{(1)} \\ S_1{}^{(1)}, \mathfrak{X}_{01}{}^{(1)} \end{vmatrix} \right)$$
$$+ X_3 \left(\mathfrak{S}_3{}^{(1)} + \varepsilon\mu \begin{vmatrix} S_1{}^{(1)}, \mathfrak{X}_{01}{}^{(1)} \\ S_2{}^{(1)}, \mathfrak{X}_{02}{}^{(1)} \end{vmatrix} \right) = 0, \tag{13'}$$

where μ is an arbitrary scalar.

If $\Sigma^{(1)} \neq 0$, we find λ is infinite, so that then we may possibly have point rays and no others. For this we must make a special investigation to determine whether the normal net $\mathfrak{L} = 0$, where $L \equiv \mathfrak{X}_{01} S_1^{(1)} + \mathfrak{X}_{02} S_2^{(1)} + \mathfrak{X}_{03} S_3^{(1)}$, has point-rays in common with $\mathfrak{S} = 0$ or not.

For the point rays of $\mathfrak{L} = 0$, we have

$$14) \qquad \mathfrak{X}_{(11)} : \mathfrak{X}_{(22)} : \mathfrak{X}_{33)} = S_1^{(1)} : S_2^{(1)} : S_3^{(1)} .$$

From the theory of the Conic Section, we know that $S_1^{(1)} : S_2^{(1)} : S_3^{(1)}$ satisfies the equation $\overline{S} = 0$ if $S^{(1)} = 0$, and therefore from § 9, this point-ray belongs to the congruence. It must be counted twice because l is a double root of the scalar part of 10).

For the case where all the first minors of $S\varDelta$ vanish, we remark that $S\,\mathrm{D}$ is always zero since $S \equiv k_i S^2_i$ and $P_{12} \equiv k_i S_i^{(1)} S_i^{(2)}$, and $S^{(1)} = 0$ since $S_i^{(1)} = 0$ $(i = 1, 2, 3)$. As before, we have the possibility of point-rays and proper rays. Again, from § 9, we saw that the field of point-rays was always included in the configuration in this case; but from the present discussion we conclude further that it must be counted twice.

We have also

$$\Sigma^{(1)} \equiv 2\,(k_i\, S_i^{(1)} S_i^{(1)*} + \Omega^{(1)}) = 2\,\Omega^{(1)}$$

where $S_i{}^*$ indicates that $\mathfrak{X}_{jk}$ has been substituted for $\mathfrak{X}_{oi}$ in S_i.

If $\Omega^{(1)} = 0$, whether on account of $\Omega \equiv 0$ or not, we see that X_1 is a ray of the congruence and that the normal net has an infinite number of parallel rays in common with $\mathfrak{S} = 0$, whatever X_2 may be. Their coordinates may be obtained as before in 12).

If $\Omega \equiv 0$ and $S^{(1)} = 0$ and in addition $S^{(2)} = 0$, equation 13) becomes an identity, since then both X_1 and X_2 belong to the congruence and $\mathfrak{l}_2 : \mathfrak{l}_1$ is arbitrary. Then all the rays of the net belong to the congruence. This case belongs to the type $X_1^2 = 0$, and, as we shall see later (§ 25) is exceptional.

If $\Omega \neq 0$, we obtain only a point ray to be counted twice.

In the preceding discussion, we have proved the following theorem, in addition to the one stated at the beginning of it.

If X_1 is a proper ray belonging to the dual conical congruence $\mathfrak{S} = 0$, the normal net

$$\mathfrak{P} \equiv X_1\, \mathfrak{S}_1^{(1)} + X_2\, \mathfrak{S}_2^{(1)} + X_3\, \mathfrak{S}_3^{(1)} = 0$$

and $\mathfrak{S} = 0$ *have an infinite number of parallel rays in common.*

This includes as a special case the theorem, previously proved, concerning the distribution of the rays in parallel pencils.

§ 19. Reciprocal dual Conical Congruences.

Now if we write

$$15)\qquad \mathfrak{S}_i \equiv \mathfrak{a}_{i1} X_1 + \mathfrak{a}_{i2} X_2 + \mathfrak{a}_{i3} X_3 = \mathfrak{r}\, U_i\,, \qquad (i = 1, 2, 3)\,,$$

$$15')\qquad U_1 X_1 + U_2 X_2^2 + U_3 X_3 = 0\,, \qquad (\mathfrak{a}_{ik} = \mathfrak{a}_{ik})\,,$$

we can eliminate X_i from the four equations and will then obtain an equation which must be satisfied by U_i, as follows

$$16)\qquad \overline{\mathfrak{S}} \equiv \begin{vmatrix} \mathfrak{a}_{11}, & \mathfrak{a}_{12}, & \mathfrak{a}_{13}, & U_1 \\ \mathfrak{a}_{21}, & \mathfrak{a}_{22}, & \mathfrak{a}_{23}, & U_2 \\ \mathfrak{a}_{31}, & \mathfrak{a}_{32}, & \mathfrak{a}_{33}, & U_3 \\ U_1, & U_2, & U_3, & 0 \end{vmatrix} = 0, \qquad \text{where } \mathfrak{a}_{ik} = \mathfrak{a}_{ki}\,;$$

which we call the reciprocal equation of $\mathfrak{S} = 0$, and the corresponding configuration is the reciprocal dual congruence to $\mathfrak{S} = 0$. For the canonical forms of the equation $\mathfrak{S} = 0$, the equation 16) takes corresponding simple forms.

If $S\varDelta \neq 0$, by means of 15), we can find an X corresponding to any U of 16), i. e. to any ray of the reciprocal congruence, we can find the corresponding ray of the original. Then we conclude that:

If $S\varDelta \neq 0$, *every proper ray of the dual conical congruence is the directrix of a normal net which has an infinite number of parallel rays in common with the reciprocal congruence.*

We will now prove the converse of this theorem, and for simplicity will use the canonical form for the equation.

When $S\varDelta \neq 0$, *if a normal net has an infinite number of parallel rays in common with the dual conical congruence, its directrix belongs to the reciprocal congruence.*

Let the equation of the congruence be

$$17_1)\qquad \mathfrak{a}_{11} X_1^2 + \mathfrak{a}_{22} X_2^2 + \mathfrak{a}_{33} X_3^2 = 0\,,$$

and that of the normal net

$$17_2)\qquad \mathfrak{a}_1 X_1 + \mathfrak{a}_2 X_2 + \mathfrak{a}_3 X_3 = 0$$

The common rays must satisfy the two sets of equations

$$18)\quad \begin{cases} a_{11}\,\mathfrak{X}_{01}^2 + a_{22}\,\mathfrak{X}_{02}^2 + a_{33}\,\mathfrak{X}_{03}^2 = 0\,, \\ a_1\,\mathfrak{X}_{01} + a_2\,\mathfrak{X}_{02} + a_3\,\mathfrak{X}_{03} = 0\,; \end{cases} \qquad \text{and}$$

$$19)\quad \begin{cases} a_{11}\,\mathfrak{X}_{01}\,\mathfrak{X}_{23} + a_{22}\,\mathfrak{X}_{02}\,\mathfrak{X}_{31} + a_{33}\,\mathfrak{X}_{03}\,\mathfrak{X}_{12} + \Omega = 0\,, \\ a_1\,\mathfrak{X}_{23} + a_2\,\mathfrak{X}_{31} + a_3\,\mathfrak{X}_{12} + \omega = 0\,. \end{cases}$$

Each of the equations 19) is satisfied by ∞^2 values for $\mathfrak{X}_{jk}$, but when each is taken in connection with the corresponding equation of 18) and the dual homogeneity also considered, we find but ∞^1 rays for any permissible values $\mathfrak{X}_{oi}$. Under the condition that 17_1) and 17_2) shall have an infinite number of parallel rays in common, the values of $\mathfrak{X}_{jk}$ corresponding to those of $\mathfrak{X}_{oi}$ obtained from 18) must satisfy 19). Therefore

$$20)\quad a_{11}\mathfrak{X}_{01} : a_1 = a_{22}\mathfrak{X}_{02} : a_2 = a_{33}\mathfrak{X}_{03} : a_3 = \Omega : a_1\mathfrak{X}_{01} + a_2\mathfrak{X}_{02} + a_3\mathfrak{X}_{03} = k.$$

From the first equation of 18), it follows that

$$21)\qquad \frac{a_1^2}{a_{11}} + \frac{a_2^2}{a_{22}} + \frac{a_3^2}{a_{33}} = 0.$$

By substituting in the last equation of 20) values of *k* properly chosen among the first three of 20), we obtain

$$\Omega - \frac{a_1\,a_{11}\,\mathfrak{X}_{01}^2}{a_1} - \frac{a_2\,a_{22}\,\mathfrak{X}_{02}^2}{a_2} - \frac{a_3\,a_{33}\,\mathfrak{X}_{03}^2}{a_3} = 0\,.$$

Then substituting for $\mathfrak{X}_{oi}$ in terms of a_i from the first three ratios of 20), we have

$$22)\qquad -\frac{2\,a_1\,a_1}{a_{11}} - \frac{2\,a_2\,a_2}{a_{22}} - \frac{2\,a_3\,a_3}{a_{33}} + \frac{a_{11}\,a_1^2}{a^2_{11}} + \frac{a_{22}\,a_2^2}{a^2_{22}} + \frac{a_{33}\,a_3^2}{a^2_{33}} = 0\,.$$

Equations 21) and 22) are, however, the conditions that the ray whose coordinates are $\mathfrak{a}_1 : \mathfrak{a}_2 : \mathfrak{a}_3$ shall belong to the congruence

$$\frac{U_1^2}{a_{11}} + \frac{U_2^2}{a_{22}} + \frac{U_3^2}{a_{33}} = 0,$$

which is the reciprocal of 17)

From these considerations, we see that for $S\Delta \neq 0$, the congruence and its reciprocal supplement each other in a way exactly analogous to that which occurs in the theory of the Conic Section. Also when $S\Delta = 0$, we expect the reciprocal relation to break down in a very similar way.

§ 20. Distribution of rays in Parallel Pencils.

We will now prove two theorems concerning the distribution of the parallel pencils in the congruences; one theorem for each of the two principal classes I and II.

First: *When $\Delta \neq 0$, to every permissible direction corresponds a single definite parallel pencil.*

We will prove this by proving that there is but one point-ray for every such direction: since from the preceding discussion (§§ 17, 18), we can conclude that if there is more than one, then the pencils are not determinate, and conversely.

We will use the equations in coordinates of the second kind and also the properties of the expressions in dual coordinates.

$$S = 0, \tag{23}$$

$$\left\{\begin{aligned} -S_3 X_{22} + S_2 X_{33} + X_1 \Omega &= 0, \\ S_3 X_{11} \qquad\qquad - S_1 X_{33} + X_2 \Omega &= 0, \\ -S_2 X_{11} + S_1 X_{22} \qquad\qquad + X_3 \Omega &= 0. \end{aligned}\right. \tag{24}$$

In these equations 24), we substitute $\mathfrak{X}'_{ii} + \lambda \mathfrak{X}_{(ii)}$ for $\mathfrak{X}_{ii}$, where $\mathfrak{X}'_{ii}$ represents the $\mathfrak{X}_{ii}$-coordinates of a definite, proper, but otherwise arbitrary, ray corresponding to a set of values $\mathfrak{X}_i$ satisfying 23), and we find that $\mathfrak{X}_{(ii)}$ must satisfy the equations

$$\left\|\begin{matrix} S_1, & S_2, & S_3 \\ \mathfrak{X}_{(11)}, & \mathfrak{X}_{(22)}, & \mathfrak{X}_{(33)} \end{matrix}\right\| = 0. \tag{25}$$

These give perfectly definite values for $\mathfrak{X}_{(ii)}$ unless $S_1 = S_2 = S_3 = 0$:

If $S\Delta \neq 0$, this can not occur.

If $S\Delta = 0$, this may occur for those values of $\mathfrak{X}_i$ for which both factors of $S = L_1 L_2$ vanish simultaneously. In this case, Ω must also vanish for this set of values, and therefore it has the form

$$2\,\Omega \equiv \lambda\, L_1 L_2 + (\lambda_2 L_1 + \lambda_1 L_2)\, L_3\,,$$

where L_3 is a linear homogeneous expression in $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$. Returning to the expression in dual coordinates, we find that $\mathfrak{S}$ can be written

$$\mathfrak{S} \equiv (l_{11} X_1 + l_{12} X_2 + l_{13} X_3)\,(l_{21} X_1 + l_{22} X_2 + l_{23} X_3)\,(1 + \lambda\,\varepsilon).$$

If this be so, we have not only $S\Delta = 0$ but also $V\Delta = 0$, and this is excluded by our hypothesis. Then, in case I, B, *i e* $S\Delta = 0$, there are no proper rays for which both factors of S vanish. Therefore our theorem is proved.

Second: If $\Delta = 0$, *and not all first minors of* $S\Delta$ *are zero, there is one direction for which the rays do not lie in entirely definite parallel pencils; if all first minors are zero, there are no entirely definite parallel pencils.*

In the preceding discussion, when $S\Delta = 0$, we found but one direction for which we could possibly have an indeterminate arrangement of parallel pencils, andthat this could happen if and only if $\mathfrak{S}$ could be written as the product of two linear homogeneous factors, *i. e.* $\Delta = 0$. This proves the first part of the statement.

If all the first minors of $S\Delta$ are zero, we know that $S_1 = S_2 = S_3 = 0$ on account of $S = 0$, and that there is no other restriction upon the values of $\mathfrak{X}_{(ii)}$ than that

$$\mathfrak{X}_1 \mathfrak{X}_{(11)} + \mathfrak{X}_2 \mathfrak{X}_{(22)} + \mathfrak{X}_3 \mathfrak{X}_{(33)} = 0\,;$$

and therefore the rays are arranged in an infinite number of parallel pencils, for which the point-rays lie in directions perpendicular to the rays considered. Our proof is then complete.

Introducing in 24), the values for S_i in terms of $\mathfrak{X}_{(ii)}$ as obtained from 25), we find that the values of Ω so obtained are exactly those which we have in equations 56) § 11 and which defined the coordinate Ω' of the parallel pencil. We can therefore use Ω as that coordinate so long as $S_i \neq 0$ $(i = 1, 2, 3)$.

VII. Geometrical Description of the Configurations of the dual Conical Congruences.

§ 21. Accessory and Absolute Congruences.

We stated early in our discussion (§ 1) that, except when

$$(\mathfrak{X}/\mathfrak{X}) = 0\,, \qquad 1)$$

there was a unique correspondence between the rays of this system and the lines of the Plücker continuum, and that when 1) was satisfied, the correspondence no longer retained that character.

In the ray geometry, equation 1) defines the so-called *Accessory Complex.*[*] In it are included congruences to which may be given the general name *Accessory Congruences,* among them, all those of class *I, A, c*) (§ 7), which we may call the *Accessory Dual Conical Congruences,* and also the whole field of point-rays. Included in these accessory congruences is one in particular, whose equation is

$$X_1^2 + X_2^2 + X_3^2 = 0,$$

and which is called the *Absolute Congruence* and which contains the so-called *Minimal-rays.*[†] These Minimal-rays and the point-rays are the only ones in this accessory complex which have any corresponding elements in the Plücker line-continuum, and in these cases the correspondence is not unique.

For the description of the congruences, we will adopt the general form of the equation given in 9) § 6, using however the general classification of 17) § 4, by not distinguishing between *a*) and *b*). We exclude class *c*) when we go to the Plücker continuum for the reasons indicated just above. We easliy pass to the equations of class *c*) by the transformation

$$X_i' = \sqrt{a_{ii}}\, X_i$$

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[*] Dynamen. pp, 286—287.

[*] Dynamen. pp. 286—287.

§ 22. Configurations of Class $I\,A$. $S\Delta \neq 0$.

We will take the equation as

2) $$\mathfrak{a}_{11} X_1^2 + \mathfrak{a}_{22} X_2^2 + \mathfrak{a}_{33} X_3^2 = 0.$$

We may then consider $S = 0$ as the equation of a cone, with the origin as vertex, whose generators determine the directions of the rays of our congruence. Taking $\mathfrak{X}_i$ $(i = 1, 2, 3)$ as proportional to the direction-cosines of a generator of the cone, we find the equation of the tangent plane along this generator to be

3) $$a_{11}\,\mathfrak{X}_1\,x_1 + a_{22}\,\mathfrak{X}_{22}\,x_2 + a_{33}\,\mathfrak{X}_{33}\,x_3 = 0.$$

From 25) § 20, it follows that the point-ray corresponding to $\mathfrak{X}_i$ has the coordinates

$$\mathfrak{X}_{(ii)} = a_{ii}\mathfrak{X}_i\,, \qquad (i = 1,\ 2,\ 3)\,,$$

and is therefore determined by the normal to this plane. Having determined one value of $\mathfrak{X}_{ii}$ $(i = 1, 2, 3)$, the parallel pencil, containing the rays given by $\mathfrak{X}_i$ is entirely determined.

Now passing to the Plücker continuum and introducing the idea of the plane containing a parallel pencil, we have from 3), 13) and 13′) § 18, that:

The plane containing a parallel pencil of a dual conical congruence of the general type coincides with that of the corresponding parallel pencil of its reciprocal and is perpendicular to the two tangent planes of the cones $S = 0$ *and* $\overline{S} = 0$, *through the corresponding rays, one in each cone.*

When $\Omega \equiv 0$, since $\mathfrak{X}_{11} = \mathfrak{X}_{22} = \mathfrak{X}_{33} = 0$ are $\mathfrak{X}_{ii}$ - coordinates of one ray of the congruence having any of the permissible directions, i. e. the ray of the cone is a ray of the congruence, we obtain a very simple method of generating the corresponding congruence: namely,

Draw those normal planes to a proper cone of the second order (i. e. $S\Delta \neq 0$*) which pass through the vertex, and in each plane construct all the rays parallel to the ray of normal intersection. The totality of rays so obtained makes up the rays of the congruence including the point-rays, which may be considered as those rays of the parallel pencils which lie at infinity.*

If Ω is not identically zero, we may consider that after the planes and parallel pencils have been determined as above, each plane with its parallel pencil is translated parallel to itself to a position determined by X_i $(i = 1, 2, 3)$ and which is given by the formula 27′) § 12.

A corresponding procedure would give the reciprocal congruence. However, after the original congruence has been determined, the reciprocal can be obtained more easily as follows:

In every plane obtained for the original dual conical congruence, construct all the rays perpendicular to those already found. The totality of new rays obtained constitute the reciprocal congruence.

§ 23. Configurations of Class *I, B, a* and *b*. $S\Delta = 0$, $\Delta \neq 0$.

For this case, the canonical form is

4) $a_{11}X_1^2 + a_{22}X_2^2 + \varepsilon(a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2) = 0,\ a_{11}a_{22}a_{33} \neq 0.$

Writing $b_1 = +\sqrt{a_{11}}$, $b_2 = +\sqrt{-a_{22}}$, we have

5) $$S \equiv L_1 L_2 \equiv (b_1\mathfrak{X}_1 - b_2\mathfrak{X}_2)(b_1\mathfrak{X}_1 + b_2\mathfrak{X}_2) = 0.$$

From this equation, we obtain the values

6) $$\mathfrak{X}_1 : \mathfrak{X}_2 : \mathfrak{X}_3 = b_2 : \pm b_1 : \varrho$$

where ϱ is an arbitrary scalar.

From 25) § 20, we have

$$\mathfrak{X}_{(11)} : \mathfrak{X}_{(22)} : \mathfrak{X}_{(33)} = S_1 : S_2 : S_3.$$

Substituting the values of $\mathfrak{X}_i$ from 6) in these expressions, we finally obtain the following values for the coordinates of the parallel pencils,

$$\Xi_1 = b_2,\ \Xi_2 = \pm b_1,\ \Xi_3 = \varrho,\ \Omega' = (a_{11}b_2^2 + a_{22}b_1^2 + a_{33}\varrho^2)$$

$$\Phi_1 = 2b_1^2b_2,\ \Phi_2 = \mp 2b_1b_2^2,\ \Phi_3 = 0,$$

in which we use the upper or the lower sign according as we use those values of $\mathfrak{X}_i$ which make the first factor or the second one of S vanish.

Then the rays of the congruence belong to two planar congruences, the rays of which are arranged in parallel pencils and are perpendicular respectively to the directions given by

$$7) \qquad b_1 : \mp b_2 : 0.$$

Turning to the Plücker continuum, we find that the coordinates of the planes for the parallel pencils of the two sets, respectively, are

$$8) \qquad u_0 : u_1 : u_2 : u_3 =$$
$$(a_{22} b_1{}^2 + a_{11} b_2{}^2 + a_{33} \varrho^2) : \pm 2 b_1 b_2{}^2 \varrho : 2 b_1{}^2 b_2 \varrho : \mp 2 b_1 b_2 (b_1{}^2 + b_2{}^2).$$

From 7) and 8), the planes of each set of 8) are parallel respectively to the corresponding direction given in 7). Since the coordinates in 8) are expressed in quadratic functions of the single variable ϱ, the planes must envelope two cylinders of the second class whose equations are:

$$9) \quad \left\{ \begin{array}{c} \sqrt{a_{11}}\, u_1 \pm \sqrt{-a_{22}}\, u_2 \;\, 0, \\ a_{33} (a^2{}_{11} - a^2{}_{22})\, u_1 u_2 \pm \sqrt{-a_{11}\, a_{22}}\, (a_{11} a_{22} - a_{22} a_{11})\, u_3{}^2 \\ -2\, a_{11} a_{22} (a_{11} - a_{22})\, u_0 u_3 = 0. \end{array} \right.$$

They are both parabolic cylinders.

From 8) and 9), we see that each is separately symmetrical with reference to the axis of x_3, since if $u_0 : u_1 : u_2 : u_3$ satisfy the equations so do $u_0 : - u_1 : - u_2 : u_3$ also. Then both cylinders have this axis for a common diameter of the two right sections given by the planes.

$$u_0 : u_1 : u_2 : u_3 = 0 : b_1 : \mp b_2 : 0 \text{ respectively.}$$

They are also symmetrical to each other with respect to the axes of x_1 and x_2; because when $u_0 : u_1 : u_2 : u_3$ satisfies one, $- u_0 : - u_1 : u_2 : u_3$ and $- u_0 : u_1 : - u_2 : u_3$ satisfies the other. Then we conclude that the two right sections are parabolas if $a_{11} - a_{22} \mp 0$.

We may state the result as follows:

Given two parabolic cylinders, the diameters of whose right sections are parallel, and which are also symmetrical with reference to a line at right angles to their common diameter, then the dual conical congruences of class I, B, a) contain the rays of the two cylindrical congruences, each of which is made up of those rays tangent to one of the parabolic cylinders which are also perpendicular to the generators through their points of contact and the limiting point-rays; it also contains all the point-rays *which lie in the plane perpendicular to the common principal axis, each counted twice.* We note further that the proper rays of each of these two cylindrical congruencces are such as are represented synectically in class A, β) of §§ 13–14.

If $a_{11} = a_{22}$, we may take it as unity and then 9) becomes

9′) $$u_0 : u_1 : u_2 : u_3 = (a_{22} - a_{11} + a_{33}\,\varrho^2) : \mp 2\,\varrho : 2\,i\,\varrho : 0$$

The planes are then arranged in two pencils of parallel planes, which are tangent to the absolute conic in the plane at infinity. Each plane is counted twice, as each contains two parallel pencils. These two pencils are such that the two points in the plane at infinity determined by them are harmonic conjugates with respect to the point of tangency with the absolute conic and the point determined by the lines of intersection of the two sets of parallel planes. The point-rays all lie in the chord of contact since they lie in a plane perpendicular to the lines of intersection of the two sets of planes. Then:

In class I, B, b), the point-rays lie in a definite line in the plane at infinity. The proper rays are arranged in parallel pencils which lie in two pencils of parallel planes, each of which touches the absolute conic in the plane at infinity in one or the other of the two points of intersection with the line of the point rays. In every plane, there are two parallel pencils which determine two points on the corresponding tangent which are harmonic conjugates with respect to the point of tangency and the pole of the chord of contact.

§ 24. Configurations of Class II. $\Delta = 0$.

A. First Minors of $S\,\Delta$ not all zero.

The canonical equation is

10) $$\mathfrak{a}_{11} X_1^2 + \mathfrak{a}_{22} X_2^2 = 0\,, \qquad a_{11}\,a_{22} \neq 0.$$

Placing $\mathfrak{b}_1 = +\sqrt{\mathfrak{a}_{11}}$, $\mathfrak{b}_2 = +\sqrt{-\mathfrak{a}_{22}}$, we have

$$(\mathfrak{b}_1 X_1 - \mathfrak{b}_2 X_2)\,(\mathfrak{b}_1 X_1 + \mathfrak{b}_2 X_2) = 0.$$

This is satisfied by all the rays of the two normal nets,

$$\mathfrak{b}_1 X_1 - \mathfrak{b}_2 X_2 = 0, \quad \text{and} \quad \mathfrak{b}_1 X_1 + \mathfrak{b}_2 X_2 = 0;$$

and also by all the rays which satisfy the two factors of $S = 0$ simultaneously; namely, the bundle for which

$$\mathfrak{X}_1 : \mathfrak{X}_2 : \mathfrak{X}_3 = 0 : 0 : 1,$$

each of which must be counted twice, since these are double values for $S = 0$. There are no other proper rays than those just mentioned. The point-rays satisfy the equation $\mathfrak{X}_{(33)}^2 = 0$, and are those belonging to the parallel bundle as limit rays; each is to be counted twice and so can be considered as belonging to the bundle. Therefore:

A congruence of class II, A consists of all the rays belonging to two proper normal nets with non-parallel principal axes each ray to be counted once; and the rays of the parallel bundle whose direction is given by the common ray of the two nets, along with the limiting point-rays, each ray and point-ray to be counted twice.

§ 25. Configurations of Class II. $\Delta = 0$.

B. All first minors of $S\,\Delta$ zero.

The canonical equation for all the subclasses is

11) $$X_1^2 + \varepsilon\,(a_{22}X_2^2 + a_{33}X_3^2) = 0.$$

This gives

12) $$\begin{cases} S \equiv \mathfrak{X}^2_{01} = 0, \quad 2\,\Omega \equiv a_{22}X_2^2 + a_{33}X_3^2, \\ \qquad \Sigma \equiv 2\,(\mathfrak{X}_{01}\mathfrak{X}_{23} + \Omega) = 0. \end{cases}$$

α. $\qquad a_{22}\,a_{33} \neq 0.$

From 12), we see that all the proper rays must have one of two definite directions, and since there are no further restrictions on the coordinates, they form two parallel bundles. Each bundle must be counted twice, since the values $\mathfrak{X}_{0i}$ are double solutions of $S = 0$.

β. $\qquad a_{33} = 0.$

In this case, the two directions coincide, and then the parallel bundles coincide, giving one parallel bundle, every ray of which must be counted four times.

From the equations 4) and 8) § 9, we know that the whole field of point-rays belongs to the congruences of these two cases; and from the discussion in connection with equation 14) § 18, it must be counted twice.

γ. $a_{22} = a_{33} = 0.$

In this case, we do not obtain a congruence but a complex, since the equation is satisfied by the ∞^3 rays for which $\mathfrak{X}_{01} = 0$, and is to be excluded when dealing with configurations having quadratic dual representations considered as congruences. However in order to make the discussion complete for the cases having such representations, we will look into it a little further.

The equation becomes

13) $$X^2 = 0$$

It may be considered as resulting from performing the multiplication in the equation

14) $$[X_1 - \varepsilon\,(a_{22}\,X_2 + a_{33}\,X_{33})] \cdot [X_1 + \varepsilon\,(a_{22}\,X_2 + a_{33}\,X_3)] = 0,$$

where a_{22}, a_{33} are entirely arbitrary quantities. Each factor separately represents a normal net, but they have parallel axes. Using this property we see that we obtain ∞^2 such nets satisfying 13), since a_{22}, a_{33} are arbitrary. If we consider that in case of substitution of $-a_{22}, -a_{33}$ for a_{22}, a_{33}, the resulting configuration is not to be counted a second time, then each of these normal nets is counted but once, except for $a_{22} = a_{33} = 0$, which is counted twice.

Equation 14) is satisfied by all rays for which $\mathfrak{X}_{01} = 0$, since each factor then reduces to ε multiplied by a factor. This gives the same result as stated above. This would also be true if a_{22} and a_{33} in the second factor were replaced by any other finite values whatever.

From a little different geometrical point of view, we might say that this configuration was made up of all the bundles of parallel rays which are at right angles to a given definite direction.

Excluding case γ) from our statements, we may summaaize as follows.

The congruences of class II, B consist of the rays of two parallel bundles, which are different or coincident in α) *or* β) *respectively, and the field of point-rays; each totality is to be counted twice.*

Having thus completed the geometrical description of the various classes of dual conical congruences, we would add the following distinguishing characteristic between the congruences of the two large classes I and II; namely,

No conical cougruence of class I contains a parallel bundle, while every congruence of class II contains at least one.

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